USC University of Southern California

EE 105: Introduction to Electrical Engineering

Lecture 7

Circuit Analysis 3

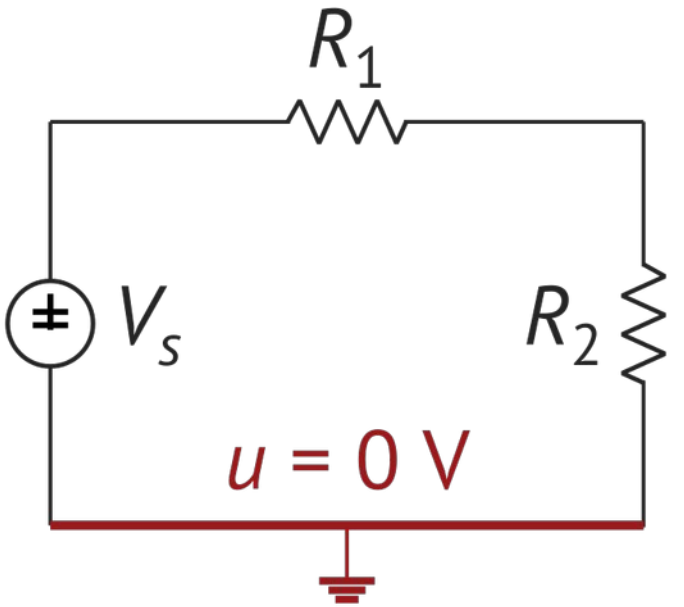
Yasser Khan

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Nodal analysis

Seven steps turn any resistor circuit into linear equations in the node voltages.

1. Pick a **reference node** (ground): u = 0 V
2. Label every other node voltage: u_1, u_2, …
3. Label a current through every element
4. Mark + and − on each element (passive sign convention)
5. Reduce the unknowns (sources, series elements)
6. Write KCL at each node and Ohm's law for each element
7. Solve the linear system

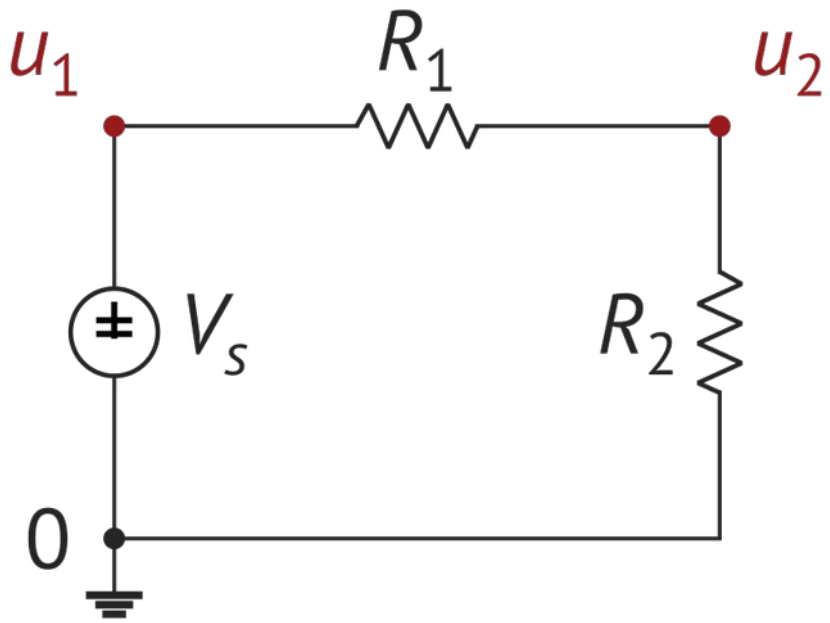


Step 1: pick a reference node

Any node works; the usual choice is the − terminal of the source. Mark it with a ground symbol; every other voltage is measured from it.

Label nodes

Step 2: name the voltage of every other node, measured from the reference node.



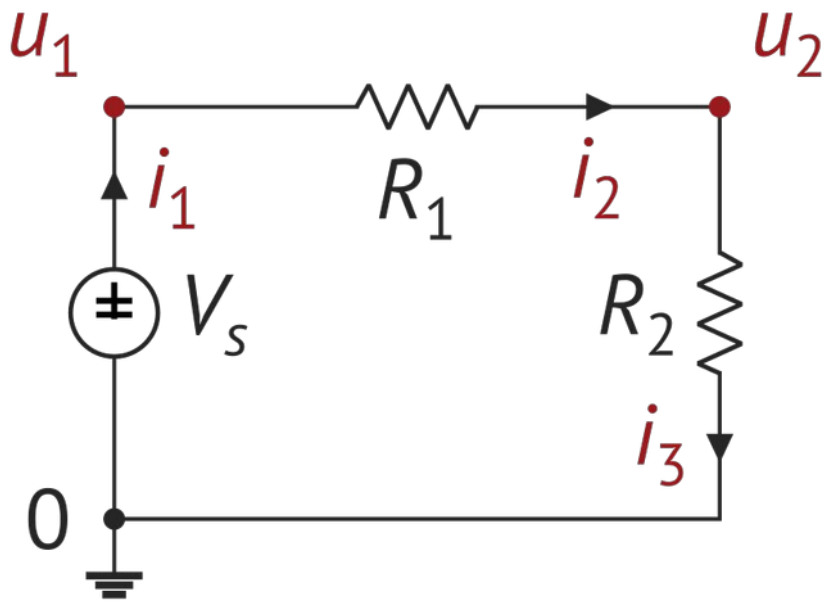
- A **node** is a whole wire between elements: this circuit has three
- The reference is 0; the others get names u_1, u_2
- u_1 sits on the + terminal of the source, so it is known
- u_2 is the unknown we want

What we know so far

$$u_1 = V_s \qquad u_2 = \ ?$$

Label currents

Step 3: put a current arrow on every element; the arrow sets the positive direction.



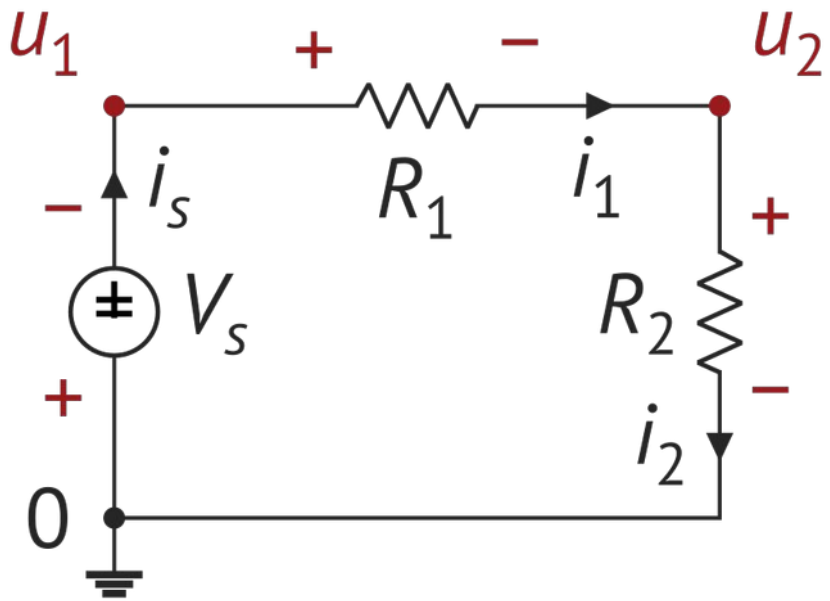
- One label per element: the current entering an element equals the current leaving it
- Pick any direction; a wrong guess just gives a negative answer
- Keep the same choice in every later step

One loop, one current

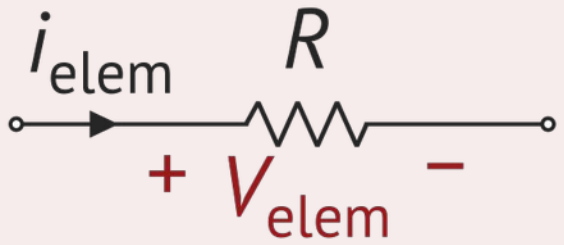
$$i_1 = i_2 = i_3$$

Voltage drop directions

Step 4: mark + and − on every element so that its current enters the + terminal.



Passive sign convention



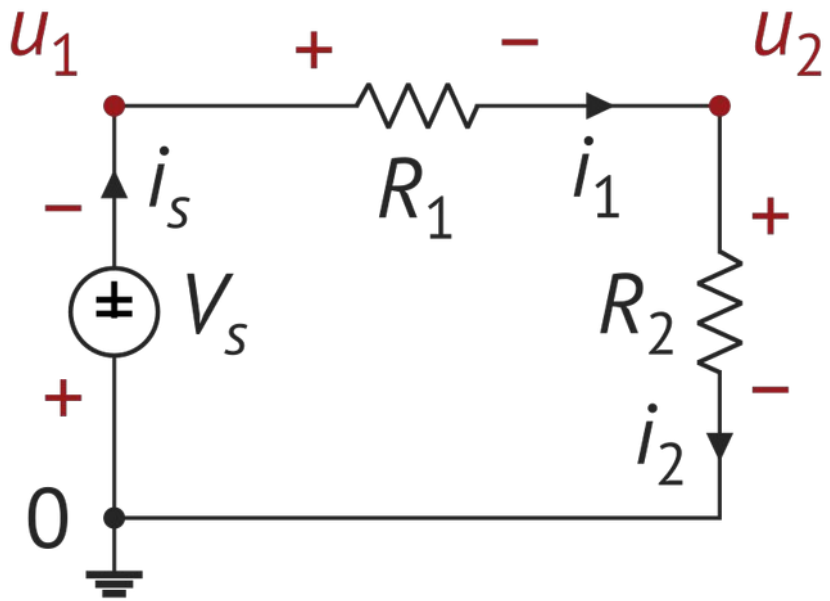
- Then V_R = i R, with no sign to remember
- i_s enters the source at the bottom, so its + mark is there

Each resistor voltage, two ways

$$V_{R1} = i_1 R_1 = u_1 - u_2 \qquad V_{R2} = i_2 R_2 = u_2 - 0$$

Write equations based on KCL and Ohm's law

Step 5: list the unknowns, then let the source and the series connections remove some.



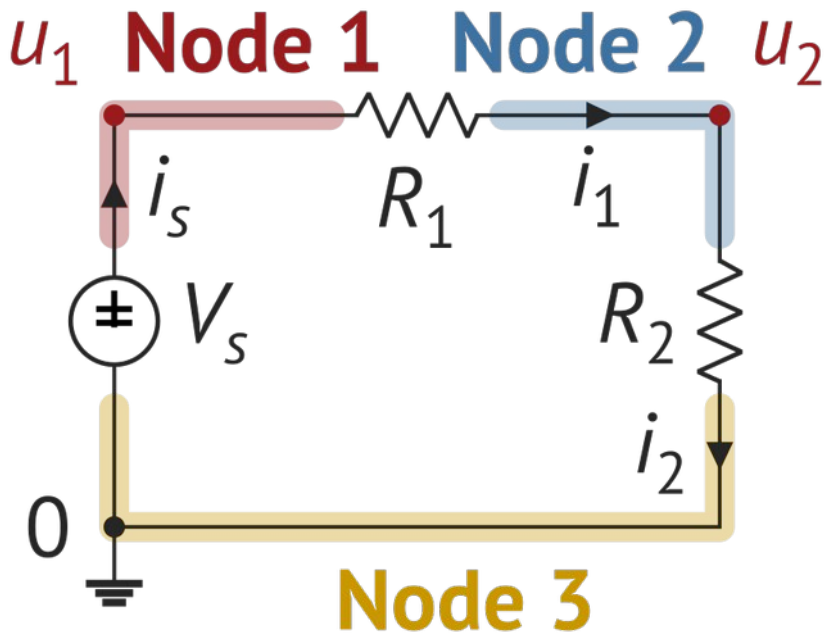
- **Unknowns**: currents i_s, i_1, i_2; node voltages u_1, u_2
- **A voltage source fixes a node**: $u_1 - 0 = V_s$
- **A node joining two elements** forces equal currents: $i_1 = i_s$, $i_2 = i_s$
- A current source would fix its branch current

Five unknowns become two

$$u_1 = V_s, \qquad i_1 = i_2 = i_s \qquad \Longrightarrow \qquad \text{unknowns: } u_2 \text{ and } i_s$$

Write equations based on KCL and Ohm's law

Step 6a: write KCL at every node: the current flowing in equals the current flowing out.



KCL, one equation per node

Node 1: $i_s - i_1 = 0$

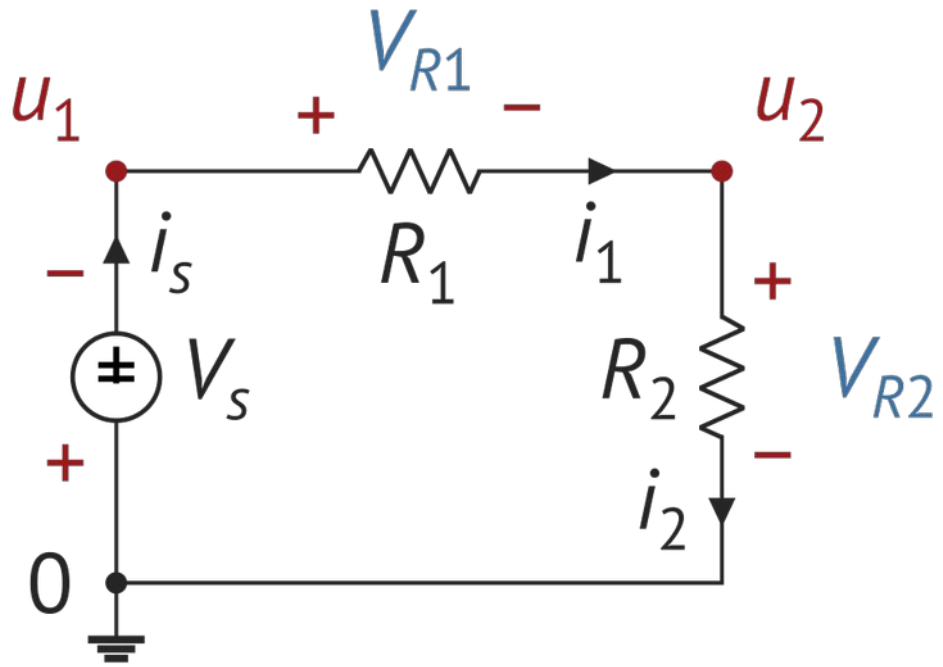
Node 2: $i_1 - i_2 = 0$

Node 3: $i_2 - i_s = 0$

- Count currents in as + and out as –
- The third equation is the sum of the first two: n nodes give n – 1 independent equations
- KCL relates the currents; Ohm's law sets their size

Write equations based on KCL and Ohm's law

Step 6b: write each element's I-V law, then express its voltage with node voltages.



Ohm's law | same voltage from node voltages

$$V_{R1} = i_1 R_1 \qquad V_{R1} = u_1 - u_2$$
$$V_{R2} = i_2 R_2 \qquad V_{R2} = u_2 - 0$$

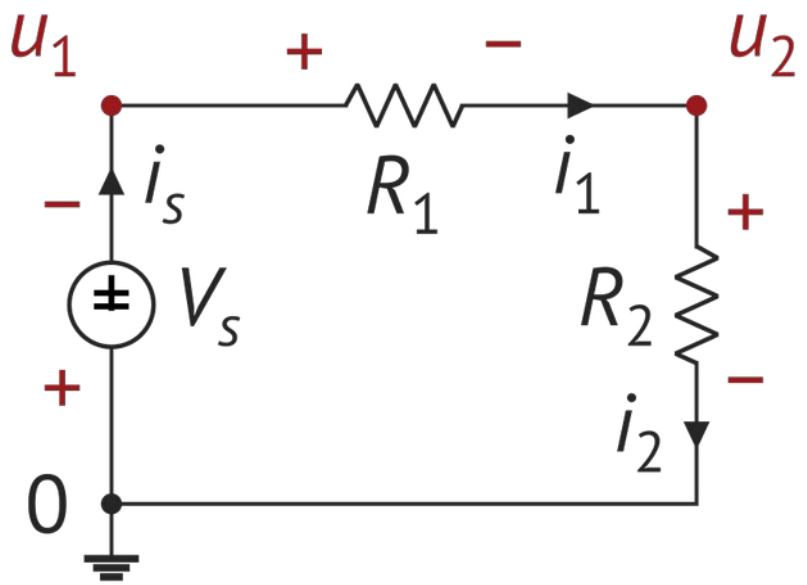
- Across the source: $u_1 - 0 = V_s$
- The + end of an element is the node written first

Combine: current through R from node a to node b

$$u_1 - u_2 = i_1 R_1 \qquad u_2 = i_2 R_2 \qquad i_{a\to b} = \frac{u_a - u_b}{R}$$

Solve for unknowns

Step 7: substitute the reductions and solve; this circuit gives back the voltage divider.



1. Substitute $u_1 = V_s$ and $i_1 = i_2 = i_s$:

$$V_s - u_2 = i_s R_1$$
$$u_2 = i_s R_2$$

2. Add them to eliminate u_2:

$$V_s = i_s (R_1 + R_2)$$

EXAMPLE

$V_s = 5$ V, $R_1 = 100\ \Omega$, $R_2 = 200\ \Omega$

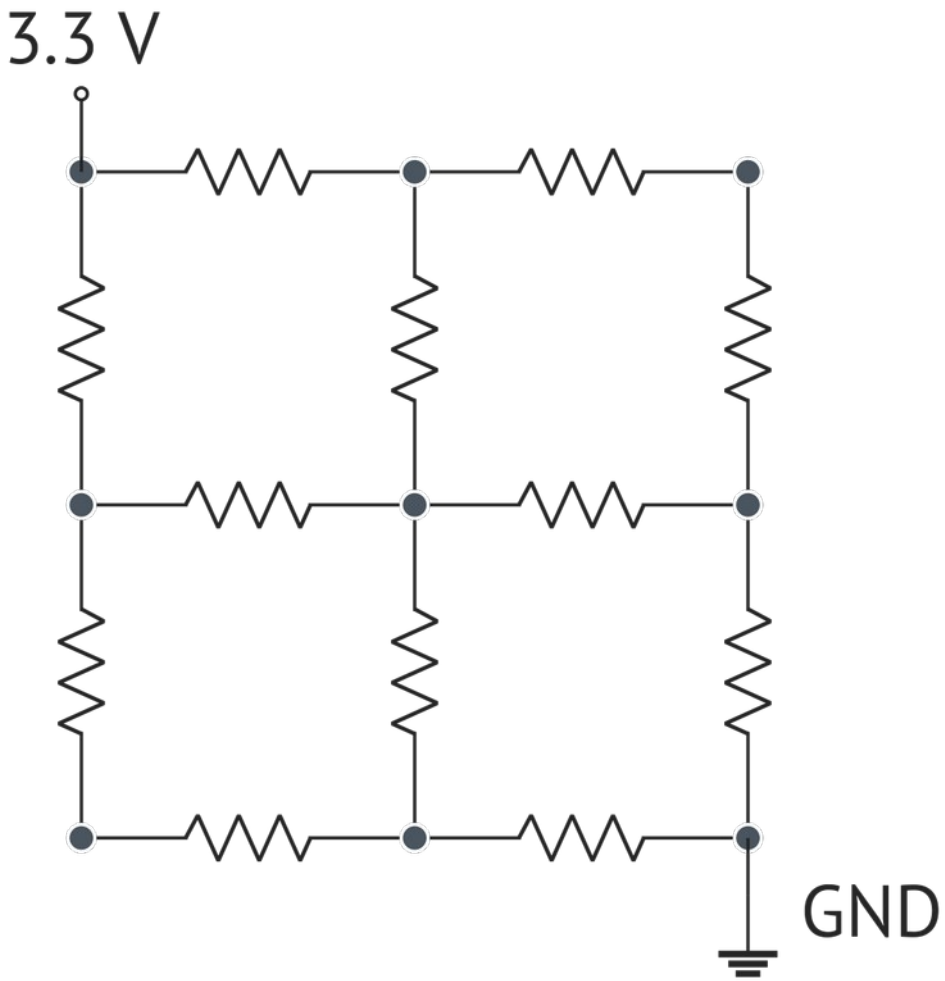
- $i_s = 5\text{ V} / 300\ \Omega =$ **16.7 mA**
- $u_2 = 5\text{ V} \times 200/300 =$ **3.33 V**, $V_{R1} = 1.67$ V

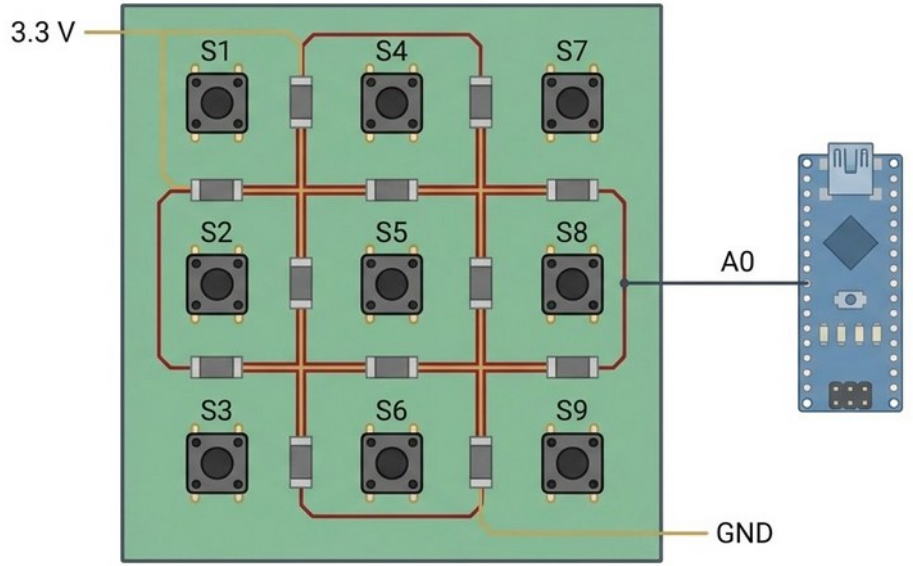
Result: the voltage divider

$$i_s = \frac{V_s}{R_1 + R_2} \qquad u_2 = \frac{R_2}{R_1 + R_2} V_s$$

Resistor grid: Network 2 on the demo board

Network 2 is the symmetric test-point grid with all twelve resistors equal; nodal analysis solves it.



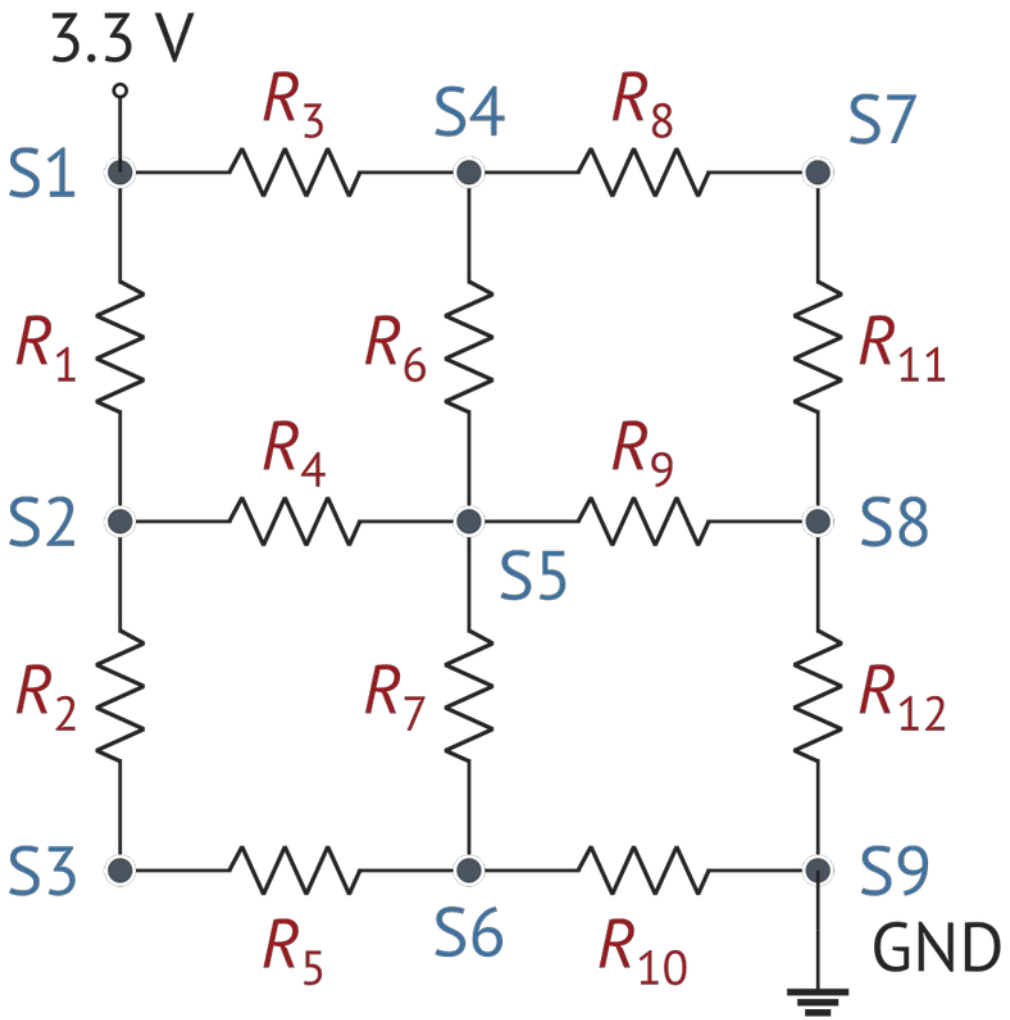


Network 1 (touch panel): buttons S1 to S9 on its nodes, sensed on pin A0 via J7.

- Twelve 100 Ω resistors; S1 on 3.3 V, S9 on ground
- No buttons: its nodes go to test points TP1 to TP9 for a multimeter
- Apart from the series pairs at S3 and S7, nothing reduces: use nodal analysis

Resistor grid: Network 2 on the demo board

Label the twelve resistors and nine nodes: S1 and S9 are known, the other seven are not.



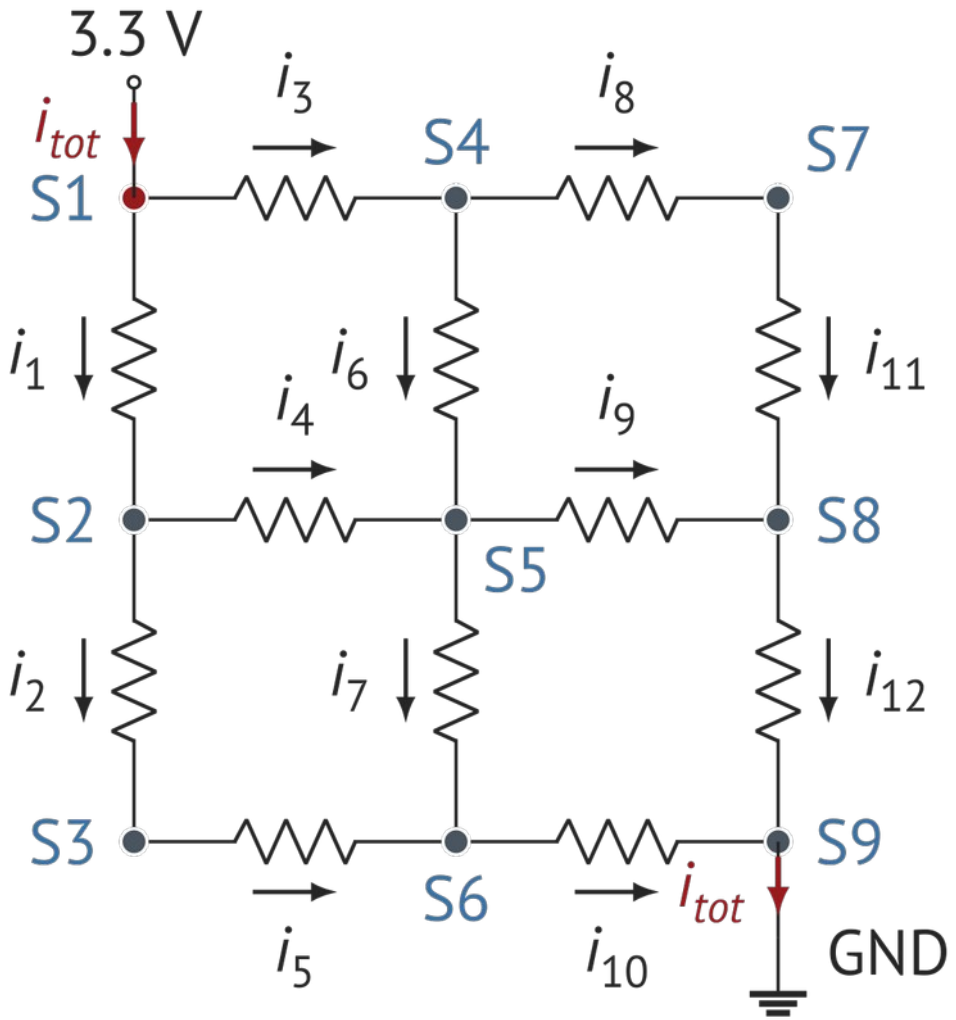
Resistor k runs from node a to node b

R_1 : S1 → S2	R_5 : S3 → S6	R_9 : S5 → S8
R_2 : S2 → S3	R_6 : S4 → S5	R_{10} : S6 → S9
R_3 : S1 → S4	R_7 : S5 → S6	R_{11} : S7 → S8
R_4 : S2 → S5	R_8 : S4 → S7	R_{12} : S8 → S9

- **Known**: S1 = 3.3 V (rail) and S9 = 0 V (ground)
- **Unknown**: S2 to S8, currents i_1 to i_{12}, and the supply current i_{tot}
- Current i_k flows through R_k from a to b, toward S9

Resistor grid: Network 2 on the demo board

KCL at each of the nine nodes: the current arriving equals the current leaving.



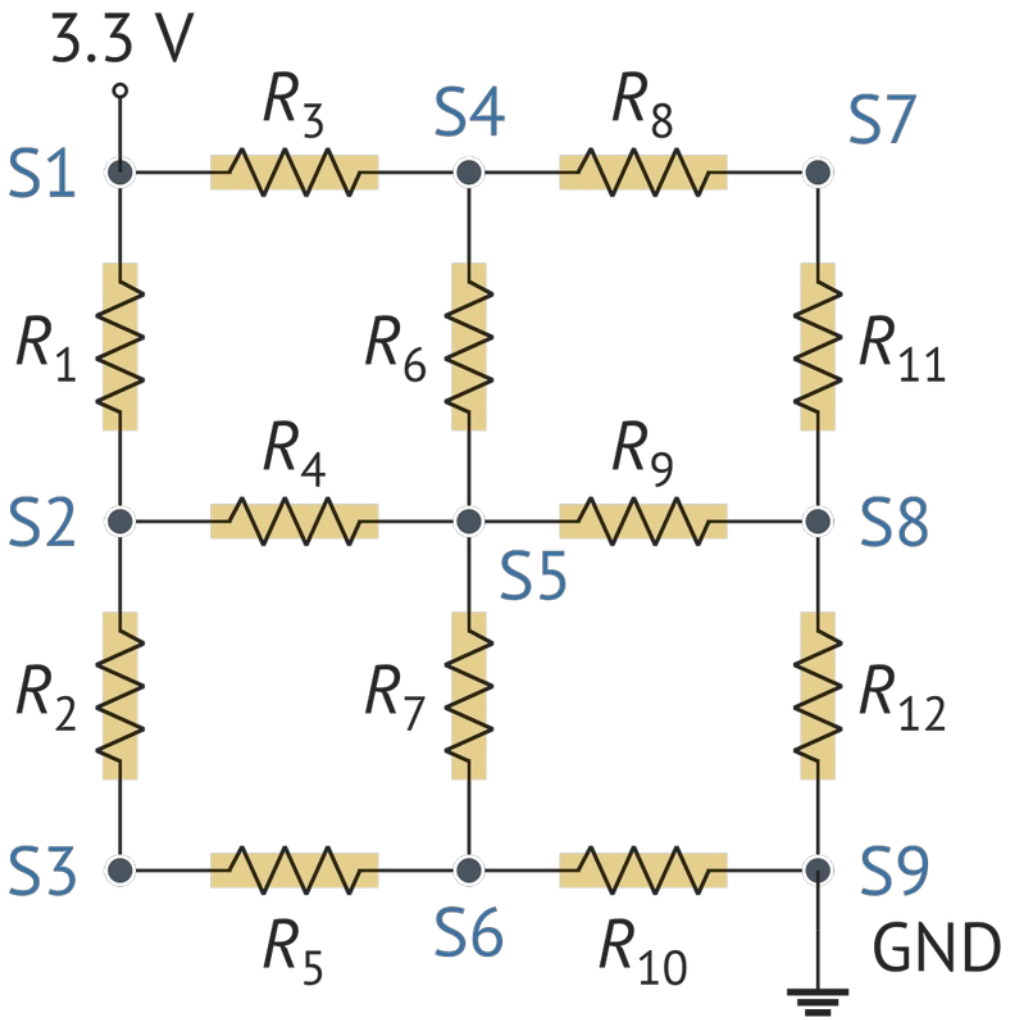
KCL at every node (in = out)

S1: $i_{tot} = i_1 + i_3$
S2: $i_1 = i_2 + i_4$
S3: $i_2 = i_5$
S4: $i_3 = i_6 + i_8$
S5: $i_4 + i_6 = i_7 + i_9$
S6: $i_5 + i_7 = i_{10}$
S7: $i_8 = i_{11}$
S8: $i_9 + i_{11} = i_{12}$
S9: $i_{10} + i_{12} = i_{tot}$

- i_{tot} enters S1 from the rail and leaves S9 to ground
- Nine nodes, nine KCL equations

Resistor grid: Network 2 on the demo board

Ohm's law for each of the twelve resistors: the voltage across it is its current times R.



One equation per resistor

$$S_1 - S_2 = i_1 R_1$$
$$S_2 - S_3 = i_2 R_2$$
$$S_1 - S_4 = i_3 R_3$$
$$S_2 - S_5 = i_4 R_4$$
$$S_3 - S_6 = i_5 R_5$$
$$S_4 - S_5 = i_6 R_6$$
$$S_5 - S_6 = i_7 R_7$$
$$S_4 - S_7 = i_8 R_8$$
$$S_5 - S_8 = i_9 R_9$$
$$S_6 - S_9 = i_{10} R_{10}$$
$$S_7 - S_8 = i_{11} R_{11}$$
$$S_8 - S_9 = i_{12} R_{12}$$

- S_k here means the voltage of node S_k
- Twelve resistors, twelve equations

Resistor grid: Network 2 on the demo board

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Nine KCL and twelve Ohm's law equations in twenty unknowns: one call to a solver.

Count before you solve

equations: 9 KCL + 12 Ohm = 21

unknowns: $i_1 \ldots i_{12}$, i_{tot}, $S_2 \ldots S_8$ = 20

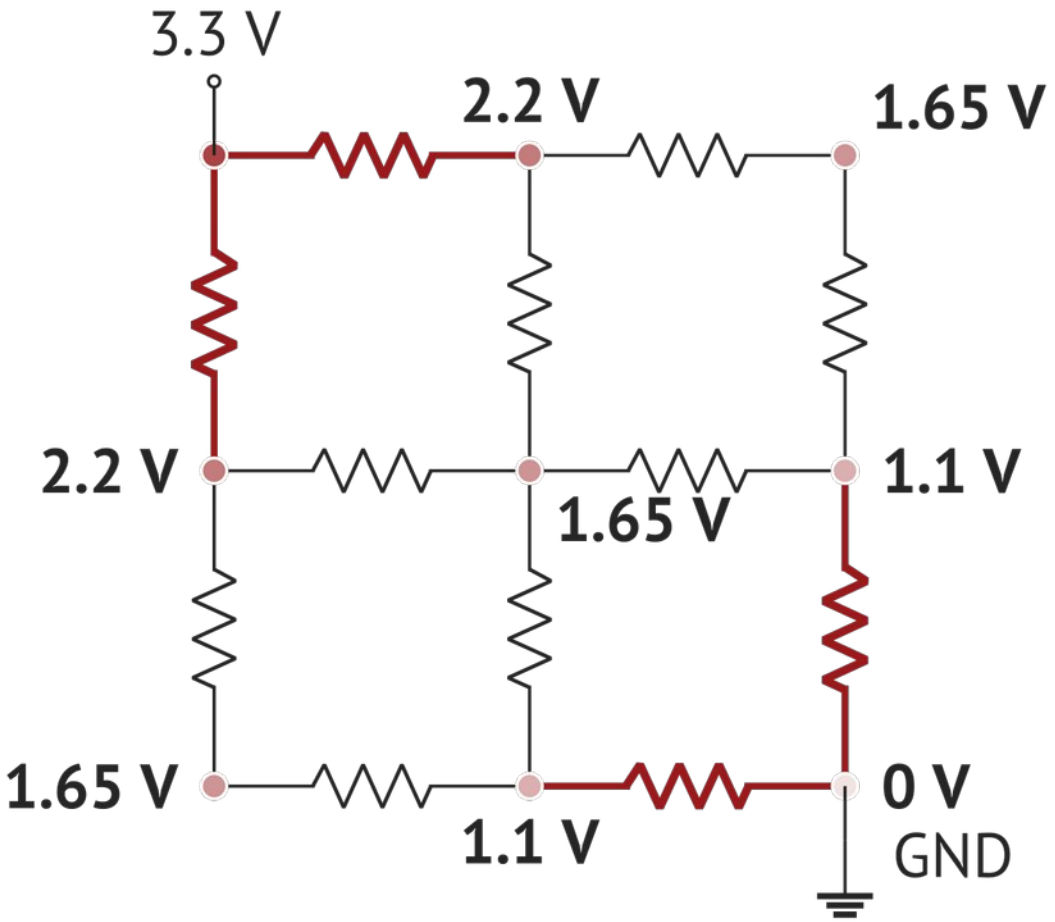
known: R_k = 100 Ω, S_1 = 3.3 V, S_9 = 0

- The nine KCL equations add up to 0 = 0, so one is redundant: 20 equations for 20 unknowns
- Too many to do by hand; SymPy solves the system (Lab 1.2)
- Enter s1 as Rational(33, 10) and SymPy returns exact fractions, e.g. s2 = 11/5

```
from sympy import symbols, Eq, solve
i = symbols('i1:13'); it = symbols('it')
s = symbols('s1:10')
r = [100]*12                    # Network 2
E = [(1,2),(2,3),(1,4),(2,5),(3,6),(4,5),
     (5,6),(4,7),(5,8),(6,9),(7,8),(8,9)]
ohm = [Eq(i[k]*r[k], s[a-1] - s[b-1])
       for k, (a, b) in enumerate(E)]
kcl = [Eq(it, i[0]+i[2]), Eq(i[0], i[1]+i[3]),
  Eq(i[2], i[5]+i[7]), Eq(i[1], i[4]),
  Eq(i[7], i[10]), Eq(i[3]+i[5], i[6]+i[8]),
  Eq(i[4]+i[6], i[9]), Eq(i[8]+i[10], i[11]),
  Eq(i[9]+i[11], it)]
known = {s[0]: 3.3, s[8]: 0}
eqs = [e.subs(known) for e in ohm + kcl]
sol = solve(eqs, [*i, it, *s[1:8]])
print(sol)
# {i1: 0.011, i2: 0.0055, ..., it: 0.022,
#  s2: 2.2, s3: 1.65, ..., s8: 1.1}
```

Resistor grid: Network 2 on the demo board

With every resistor at 100 Ω the grid settles at five voltage levels and two current values.



Node voltages (V)

S1	S2	S3	S4	S5	S6	S7	S8	S9
3.3	2.2	1.65	2.2	1.65	1.1	1.65	1.1	0

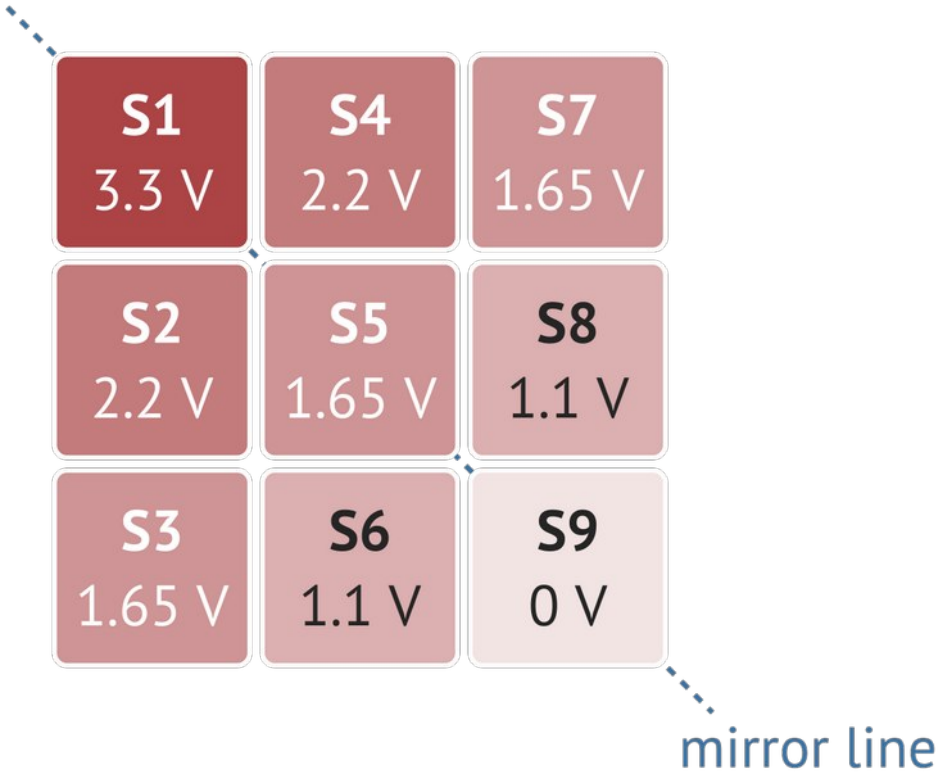
Currents (mA)

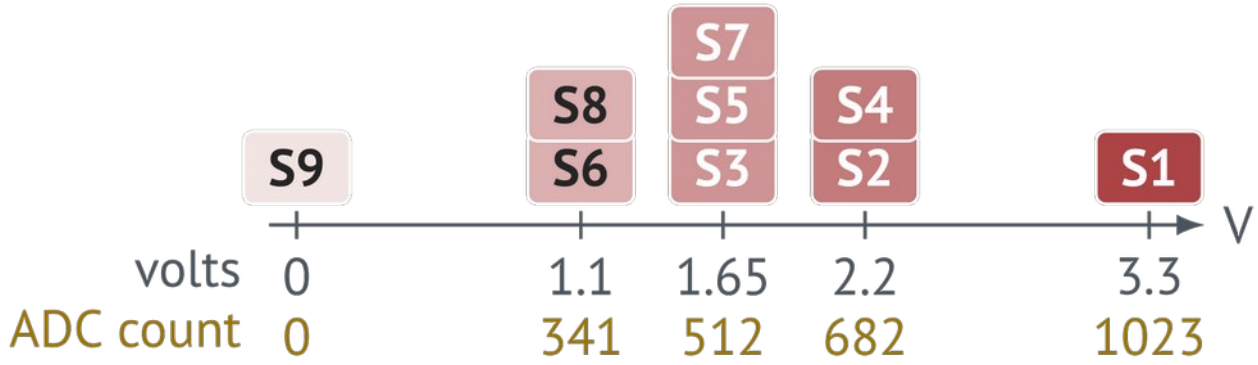
$i_1 = i_3 = i_{10} = i_{12}$ = 11
the other eight = 5.5
i_{tot} = 22

- Grid resistance seen by the rail: 3.3 V / 22 mA = **150 Ω**
- Check KCL at S5: 5.5 + 5.5 in = 5.5 + 5.5 out

Resistor grid: Network 2 on the demo board

Nine nodes but only five voltages: nodes mirrored across the S1-S9 diagonal read the same.





- ADC count N = V / 3.3 V × 1023 (10-bit)
- As buttons, S2 and S4 would both read 682; S3, S5, S7 about 512; S6 and S8 341
- Fix: make horizontal and vertical resistors differ (Network 1 uses 50 Ω rungs)

Resistor grid: Network 2 in matrix form

Keep only the seven node voltages: KCL at each node gives a 7×7 system G v = i.

G v = i (G in siemens, i in amperes)

$$\frac{1}{100}\begin{bmatrix} 3 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 4 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \end{bmatrix} = \frac{1}{100}\begin{bmatrix} 3.3 \\ 0 \\ 3.3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

G v i

- Diagonal: sum of conductances at the node (S5 has four: 4/100)
- Off-diagonal: minus the conductance joining two nodes
- Right side: V/R from known neighbors (S1 feeds S2 and S4)

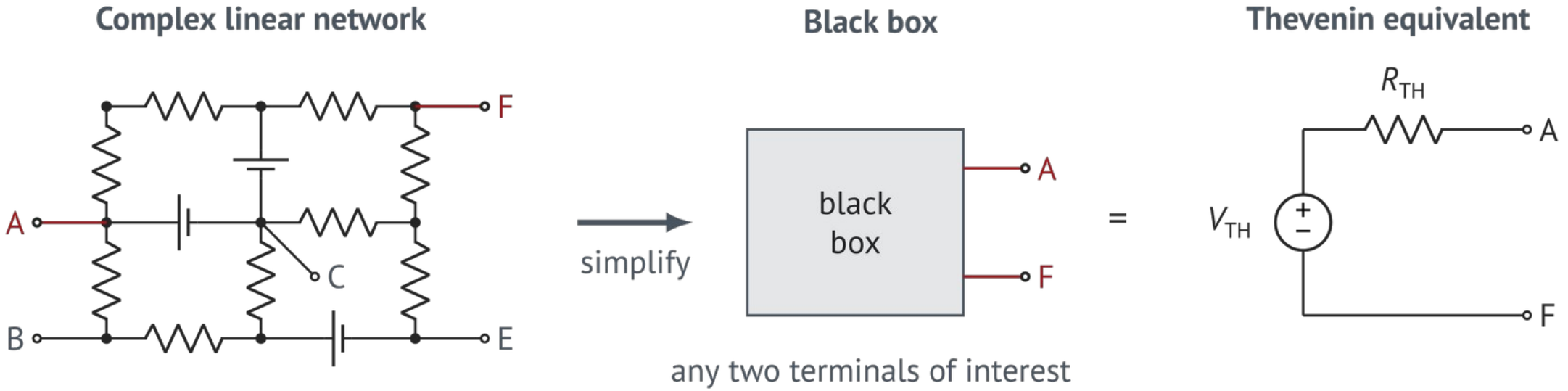
```
import numpy as np
G = np.array([[ 3,-1, 0,-1, 0, 0, 0],   # S2
              [-1, 2, 0, 0,-1, 0, 0],   # S3
              [ 0, 0, 3,-1, 0,-1, 0],   # S4
              [-1, 0,-1, 4,-1, 0,-1],   # S5
              [ 0,-1, 0,-1, 3, 0, 0],   # S6
              [ 0, 0,-1, 0, 0, 2,-1],   # S7
              [ 0, 0, 0,-1, 0,-1, 3]])  # S8
G = G / 100                             # siemens
i = np.array([3.3, 0, 3.3, 0, 0, 0, 0]) / 100
v = np.linalg.solve(G, i)
print(v.round(3))
# [2.2  1.65 2.2  1.65 1.1  1.65 1.1 ]
```

Same answer, smaller problem

A 7×7 matrix instead of 21 equations, and the same voltages as SymPy. Circuit simulators such as SPICE work this way.

Equivalent circuits

Seen from two terminals, any linear network of sources and resistors acts like one source and one resistor.



- Pick the two terminals you care about; everything else goes into the black box
- **Thevenin**: a voltage source V_{TH} in series with a resistor R_{TH}
- **Norton**: a current source I_N in parallel with $R_N = R_{TH}$
- A load on the terminals sees the same voltage and current in every version

Three numbers describe the box

$V_{TH} = V_{AF}$ with the terminals open

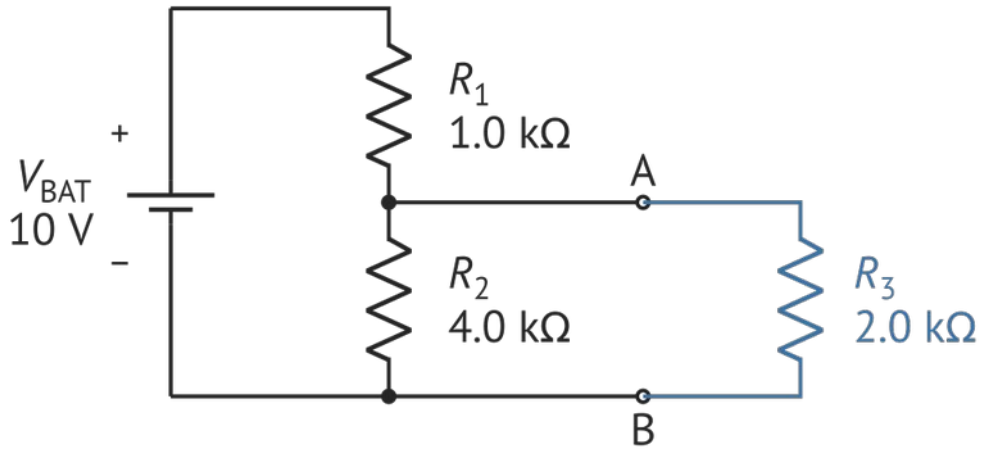
$I_N = I_{AF}$ with the terminals shorted

$R_{TH} = R_N = V_{TH}/I_N$

Thevenin's equivalent circuit

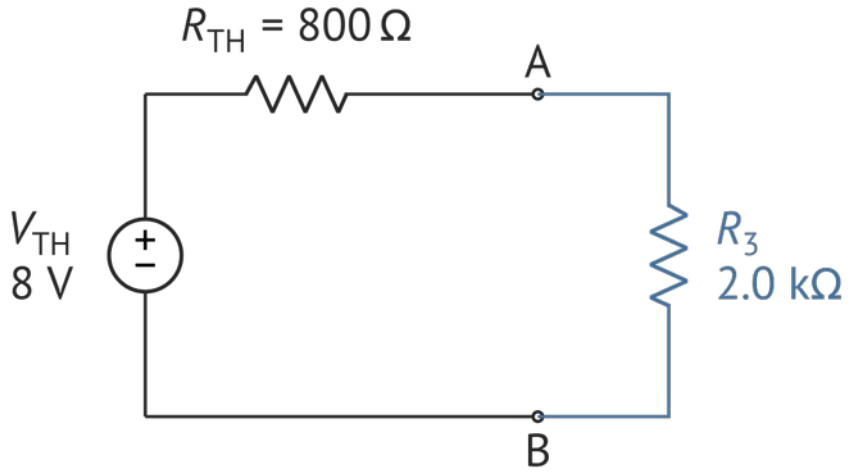
The battery and divider act like 8 V behind 800 Ω; the load R_3 cannot tell the difference.

Real circuit



=

Thevenin equivalent circuit



Finding V_{TH}

Remove the load. V_{TH} is the open-circuit voltage from A to B.

Here R_1 and R_2 form a divider: 10 V × 4/(1 + 4) = **8 V**.

Finding R_{TH}

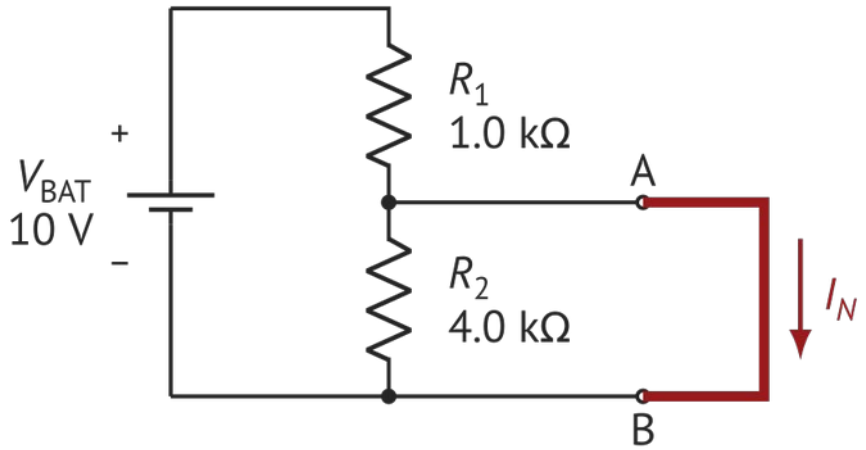
Turn the source off (the battery becomes a wire) and find the resistance between A and B.

Here $R_1 \parallel R_2$ = 1 kΩ ∥ 4 kΩ = **800 Ω**.

Norton's equivalent circuit

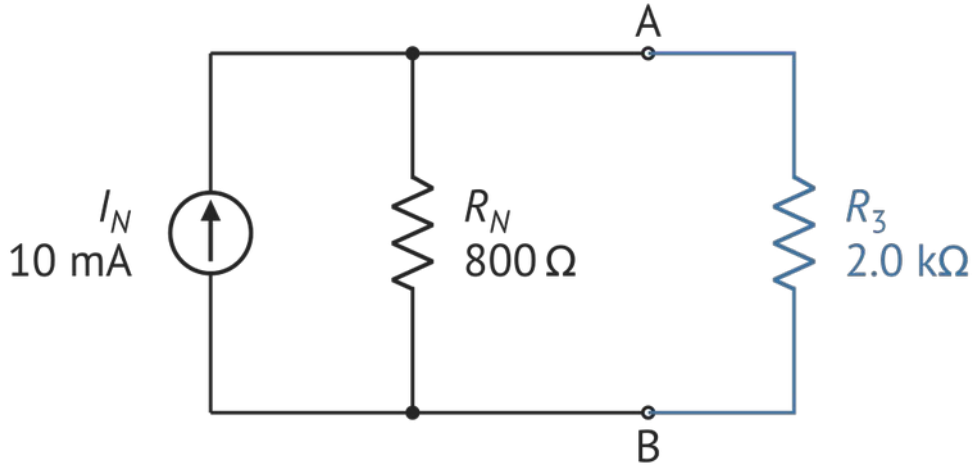
The same network is also a 10 mA source in parallel with 800 Ω; $V_{TH} = I_N R_N$ links the two forms.

Real circuit, load replaced by a short



=

Norton equivalent circuit



Finding I_N

Replace the load with a short. I_N is the current in the short.

Here: 10 V / 1 kΩ = **10 mA**.

Finding R_N

Same as R_{TH}: turn the sources off and look back into the terminals.

$R_N = R_{TH}$ = **800 Ω**.

Link between the forms

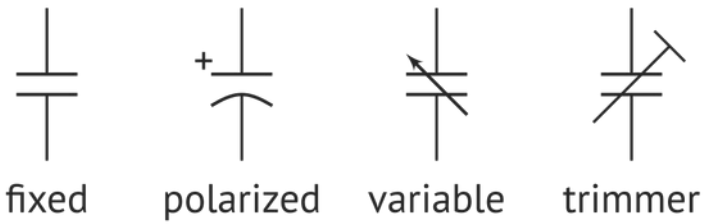
$$V_{TH} = I_N R_N$$

$$8\,\text{V} = 10\,\text{mA} \times 800\,\Omega$$

Capacitors

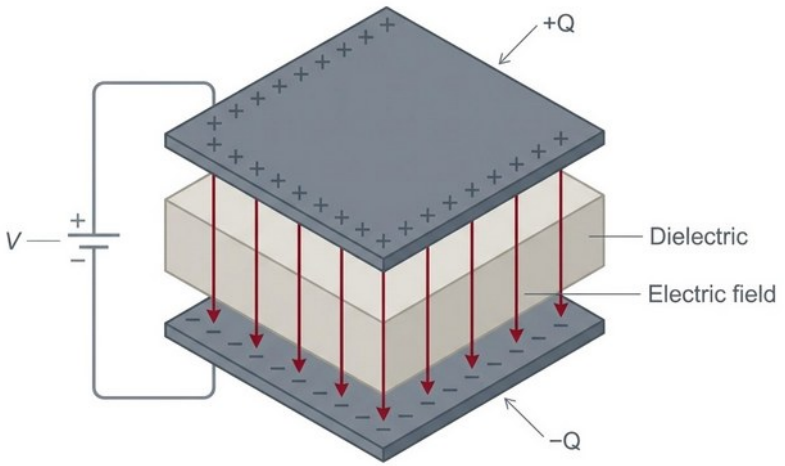
A capacitor stores charge on two conductors separated by an insulator, in proportion to the voltage.

Capacitor symbols



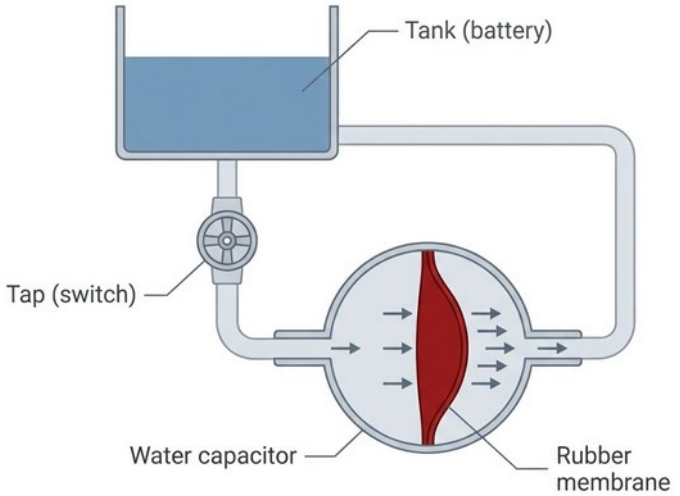
- Unit: farad (F); practical values pF to mF
- Polarized (electrolytic) parts have a + lead

Parallel-plate capacitor



+Q and −Q on the plates; the electric field in the dielectric stores the energy.

Water analogy



A rubber membrane in the pipe stretches until it pushes back as hard as the pump.

Capacitor relations

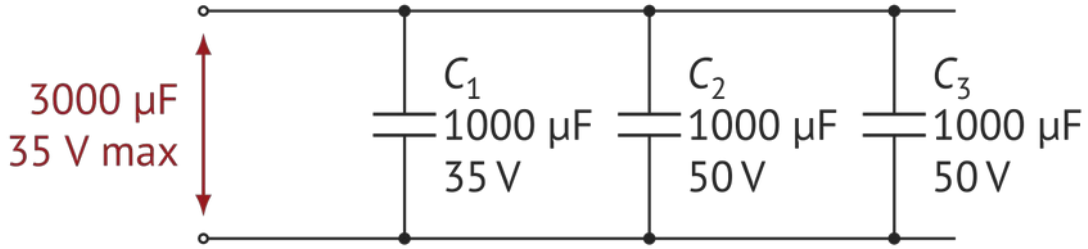
$$Q = CV \qquad C = \frac{\varepsilon A}{d} \qquad i = C\frac{dv}{dt} \qquad E = \tfrac{1}{2}CV^2$$

- Constant v: i = 0, an **open circuit** at DC
- The voltage cannot jump instantly

Capacitors in parallel and in series

Parallel capacitances add; in series their reciprocals add. These are the resistor rules swapped.

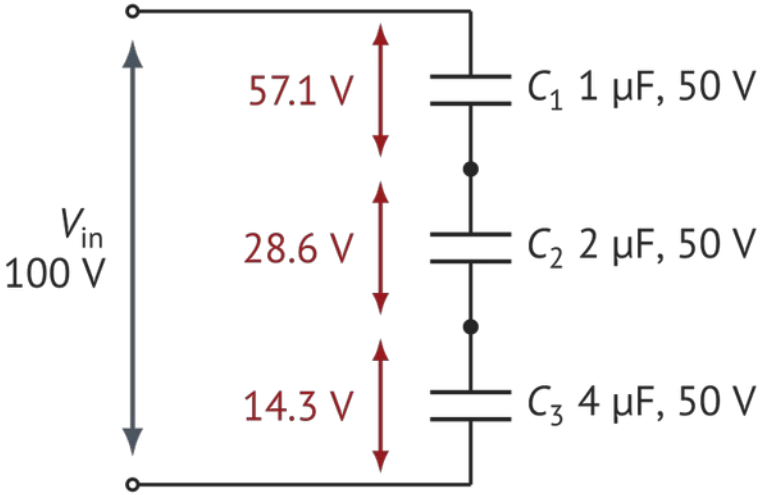
In parallel: more capacitance



$$C_{par} = C_1 + C_2 + C_3 = 3000\ \mu F$$

- Same voltage on each; the lowest rating (35 V) limits the bank

In series: higher voltage, less capacitance



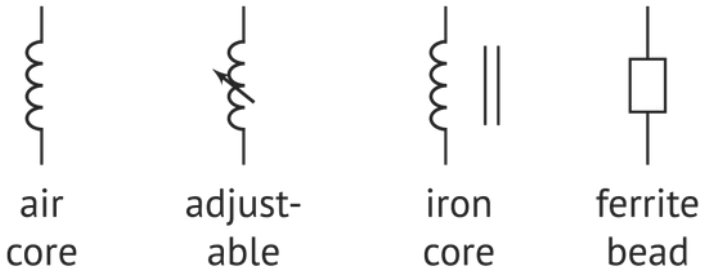
$$\frac{1}{C_{ser}} = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} \Rightarrow C_{ser} = 0.571\ \mu F$$

- Same charge (57.1 µC) on each, $V_k = Q/C_k$: the smallest C takes the most voltage
- C_1 is over its 50 V rating: the stack is safe only to 87.5 V, not 150 V

Inductors

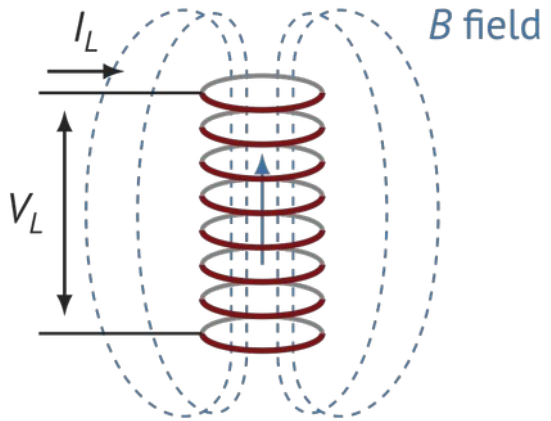
An inductor is a coil of wire; it stores energy in its magnetic field and opposes changes in current.

Inductor symbols



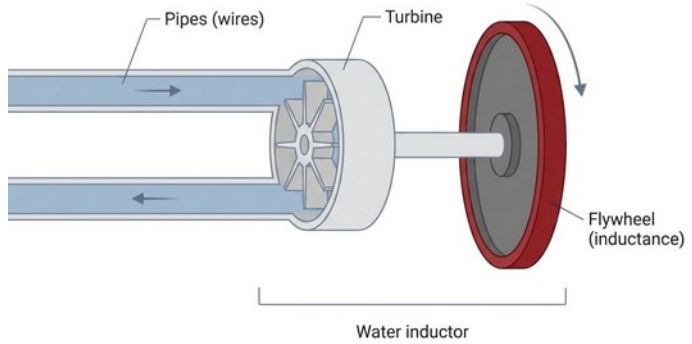
- Unit: henry (H); practical values nH to H
- An iron core raises L for the same coil

Air-core inductor



Current in the coil makes a magnetic field B; changing the current changes B, which induces V_L.

Water analogy



A turbine on a heavy flywheel resists spin-up, then keeps the water moving.

Inductor relations

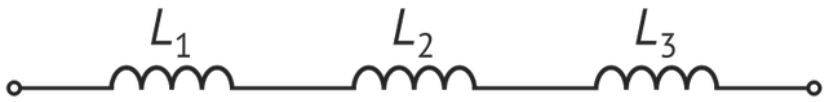
$$V_L = L\frac{dI_L}{dt} \qquad I_L = \frac{1}{L}\int V_L\,dt \qquad E = \tfrac{1}{2}LI_L^2$$

- Constant I: v = 0, a **short circuit** at DC
- The current cannot jump instantly

Inductors in series and in parallel

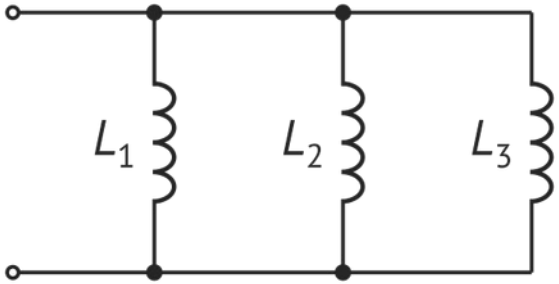
Inductors combine like resistors: series inductances add, and in parallel the reciprocals add.

In series



$$L_{ser} = L_1 + L_2 + L_3$$

In parallel



$$\frac{1}{L_{par}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

Series and parallel rules side by side

Element	Series	Parallel
Resistor R	add	reciprocals add
Inductor L	add	reciprocals add
Capacitor C	reciprocals add	add

EXAMPLE

L_1 = 1 mH, L_2 = 2 mH, L_3 = 4 mH

- Series: 1 + 2 + 4 = **7 mH**
- Parallel: 1/(1 + 1/2 + 1/4) = **0.571 mH**, below the smallest