USC University of Southern California

EE 105: Introduction to Electrical Engineering

Lecture 8

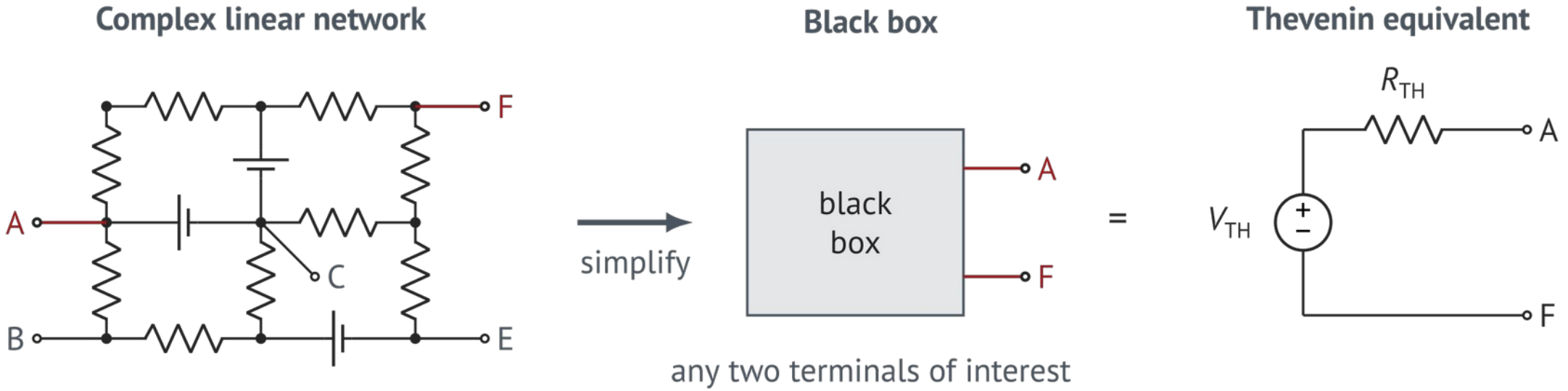
Circuit Problems: Recap

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Equivalent circuits

Seen from two terminals, any linear network of sources and resistors acts like one source and one resistor.



- Pick the two terminals you care about; everything else goes into the black box
- **Thevenin**: a voltage source V_{TH} in series with a resistor R_{TH}
- **Norton**: a current source I_N in parallel with $R_N = R_{TH}$
- A load on the terminals sees the same voltage and current in every version

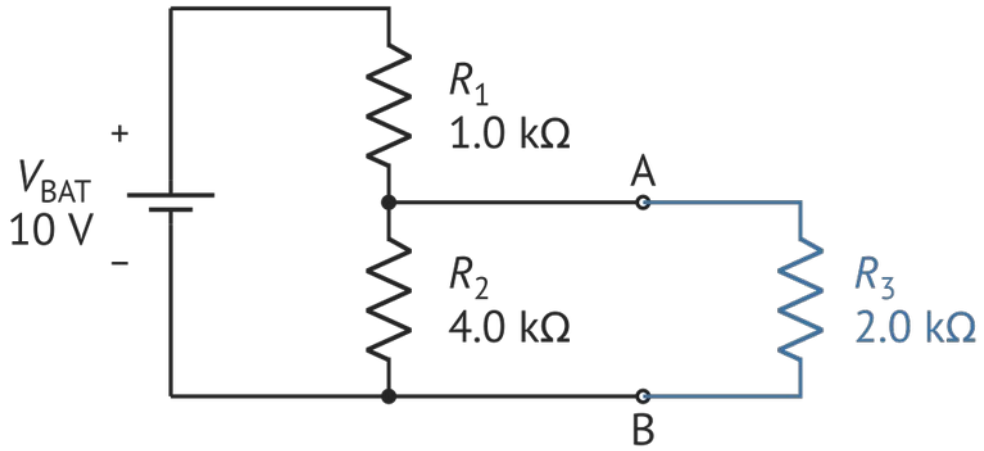
Three numbers describe the box

$$V_{TH} = V_{AF} \text{ with the terminals open}$$
$$I_N = I_{AF} \text{ with the terminals shorted}$$
$$R_{TH} = R_N = V_{TH}/I_N$$

Thevenin's equivalent circuit

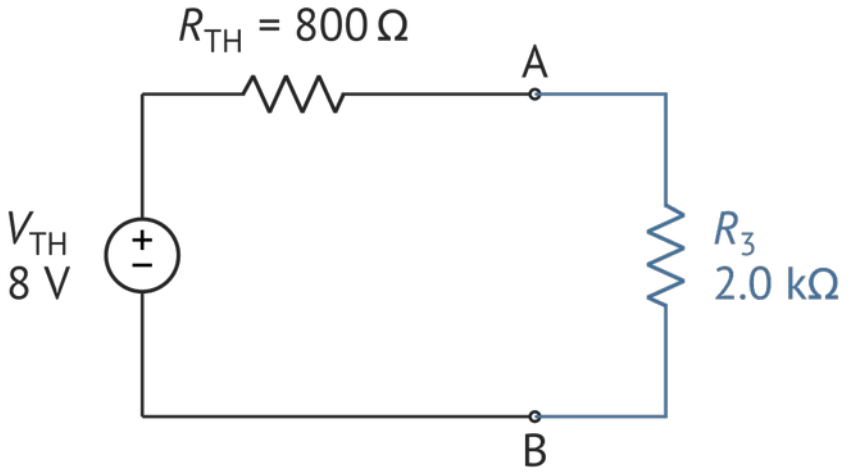
The battery and divider act like 8 V behind 800 Ω; the load R_3 cannot tell the difference.

Real circuit



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Thevenin equivalent circuit



Finding V_{TH}

Remove the load. V_{TH} is the open-circuit voltage from A to B.

Here R_1 and R_2 form a divider: 10 V × 4/(1 + 4) = **8 V**.

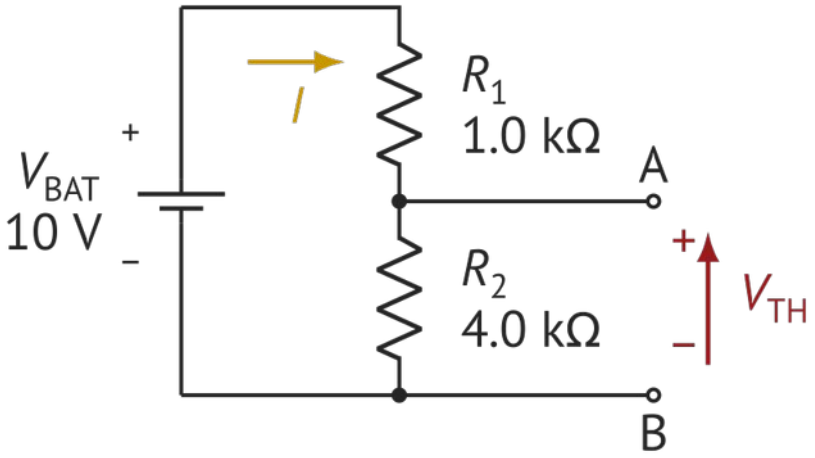
Finding R_{TH}

Turn the source off (the battery becomes a wire) and find the resistance between A and B.

Here $R_1 \parallel R_2$ = 1 kΩ ∥ 4 kΩ = **800 Ω**.

Thevenin example, step 1: open-circuit voltage

With the load removed, R_1 and R_2 form a divider, so V_{TH} is the divider output.



Load R_3 removed: terminals A and B are open.

EXAMPLE

V_{BAT} = 10 V, R_1 = 1 kΩ, R_2 = 4 kΩ, load R_3 = 2 kΩ. Find the Thevenin equivalent at A-B, then the load voltage and current.

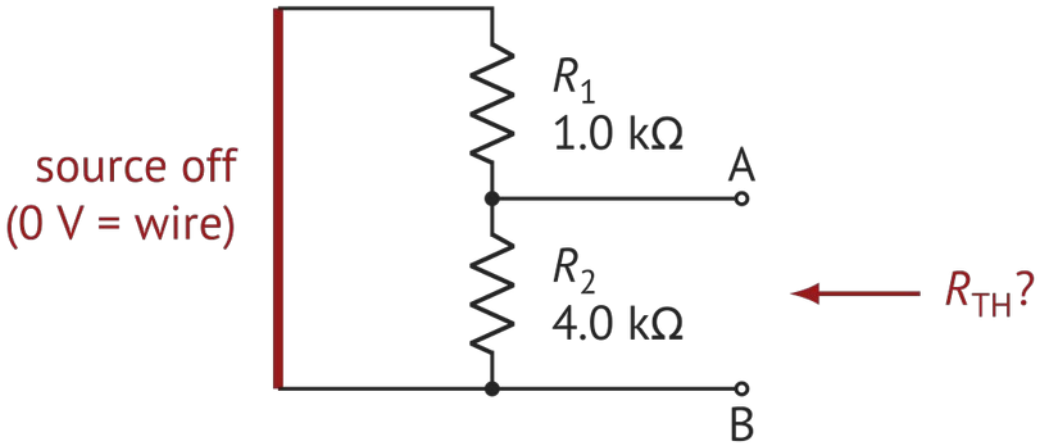
1. Remove R_3. With A-B open, no current leaves at A, so R_1 and R_2 carry the same current I
2. I = 10 V / (1 kΩ + 4 kΩ) = **2 mA**
3. $V_{TH} = V_{AB} = I R_2$ = 2 mA × 4 kΩ = **8 V**

Divider rule

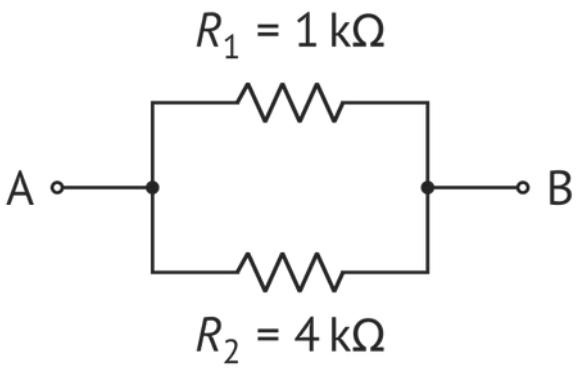
$$V_{TH} = V_{BAT}\frac{R_2}{R_1 + R_2} = 10\,\text{V} \times \frac{4}{1+4} = 8\,\text{V}$$

Thevenin example, step 2: Thevenin resistance

Turn the battery off and look back into A-B: R_1 and R_2 appear in parallel.



Redrawn: both resistors run from A to B



EXAMPLE

1. Set the 10 V source to 0 V: replace it with a wire
2. From A, one path runs through R_2 to B; the other runs through R_1 and the wire to B
3. Both resistors connect A to B, so they are in parallel
4. $1/R_{TH} = 1/(1\ k\Omega) + 1/(4\ k\Omega) = 5/(4\ k\Omega)$, so R_{TH} = **800 Ω**

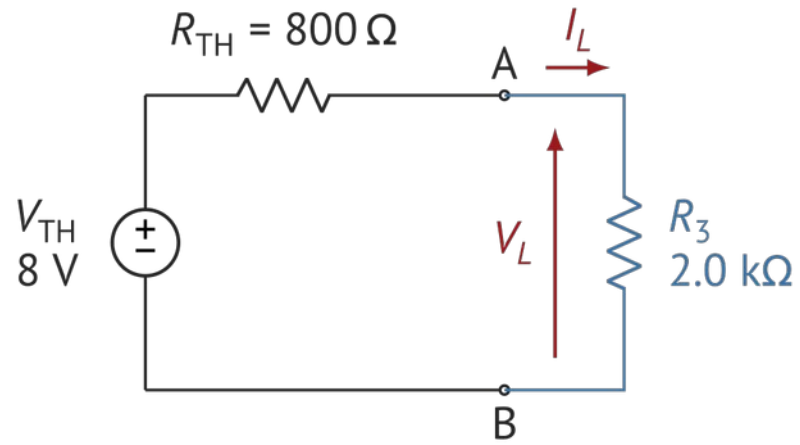
Two resistors in parallel

$$R_{TH} = \frac{R_1 R_2}{R_1 + R_2} = \frac{(1)(4)}{1+4}\ k\Omega = 800\ \Omega$$

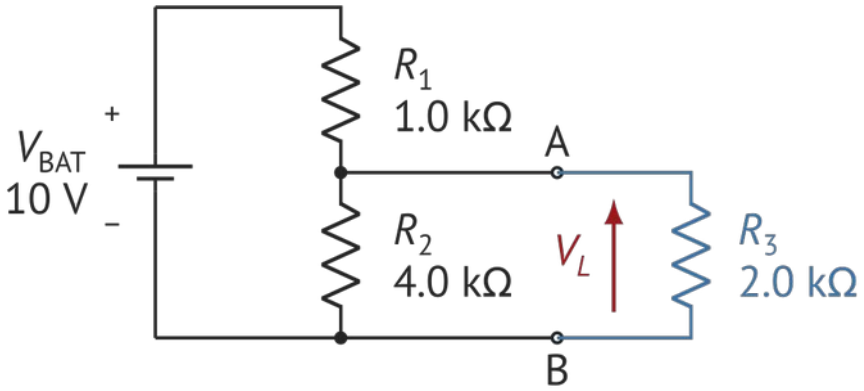
Thevenin example, step 3: attach the load

With the equivalent, the load forms a simple divider with R_{TH}: each new load takes one line.

Thevenin equivalent with the load



Check: the full circuit with the load



EXAMPLE

1. V_L = 8 V × 2/(0.8 + 2) = **5.71 V**
2. I_L = 8 V / (0.8 kΩ + 2 kΩ) = **2.86 mA**
3. Check on the full circuit: $R_2 \parallel R_3$ = 4 kΩ ‖ 2 kΩ = 1.33 kΩ, and 10 V × 1.33/(1 + 1.33) = **5.71 V**, the same

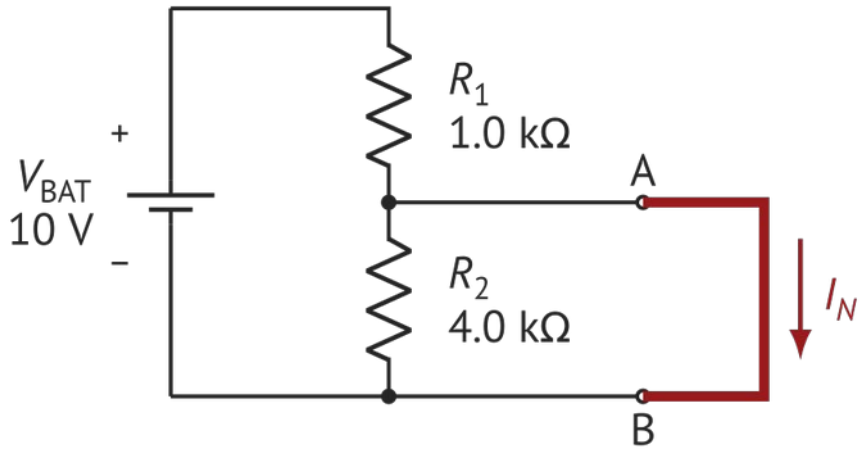
Try other loads (one line each)

R_3	1 kΩ	2 kΩ	10 kΩ	open
V_L	4.44 V	5.71 V	7.41 V	8.00 V
I_L	4.44 mA	2.86 mA	0.74 mA	0

Norton's equivalent circuit

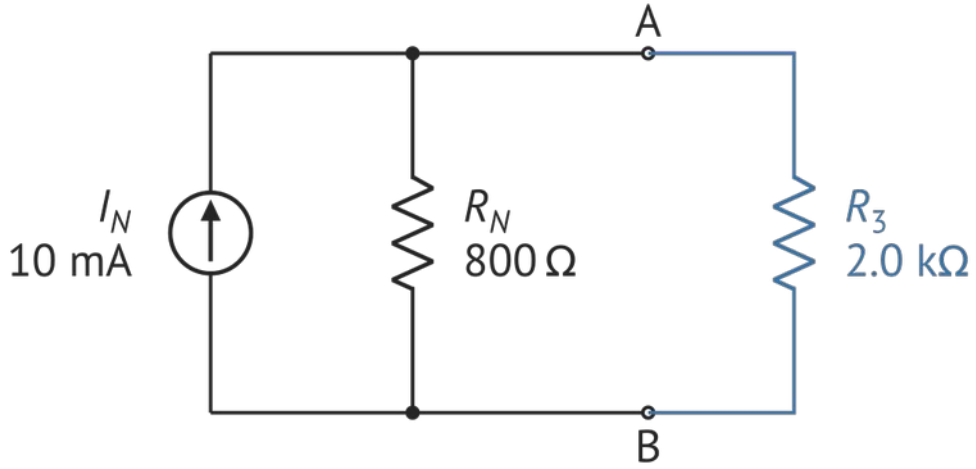
The same network is also a 10 mA source in parallel with 800 Ω; $V_{TH} = I_N R_N$ links the two forms.

Real circuit, load replaced by a short



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Norton equivalent circuit



Finding I_N

Replace the load with a short. I_N is the current in the short.

Here: 10 V / 1 kΩ = **10 mA**.

Finding R_N

Same as R_{TH}: turn the sources off and look back into the terminals.

$R_N = R_{TH}$ = **800 Ω**.

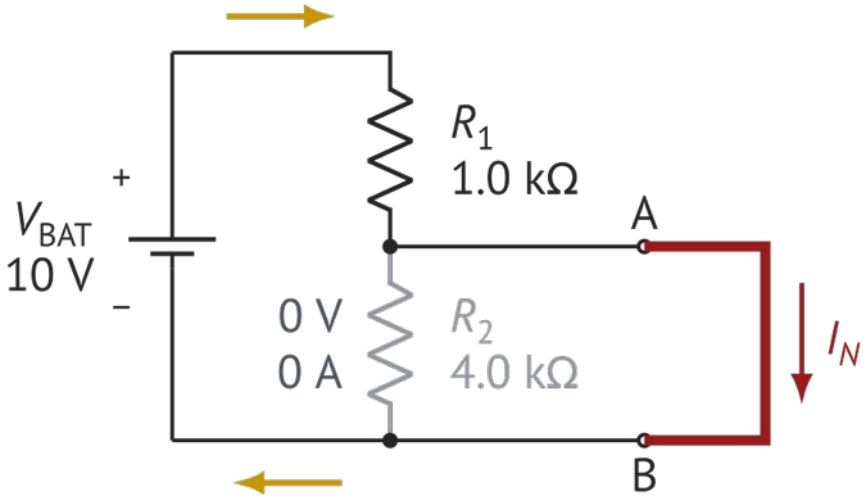
Link between the forms

$$V_{TH} = I_N R_N$$

$$8\,\text{V} = 10\,\text{mA} \times 800\,\Omega$$

Norton example: short-circuit current

Shorting A to B bypasses R_2, so the battery drives its full 10 V across R_1 alone.



Arrows show the current path; R_2 is shorted out (0 V, 0 A).

EXAMPLE

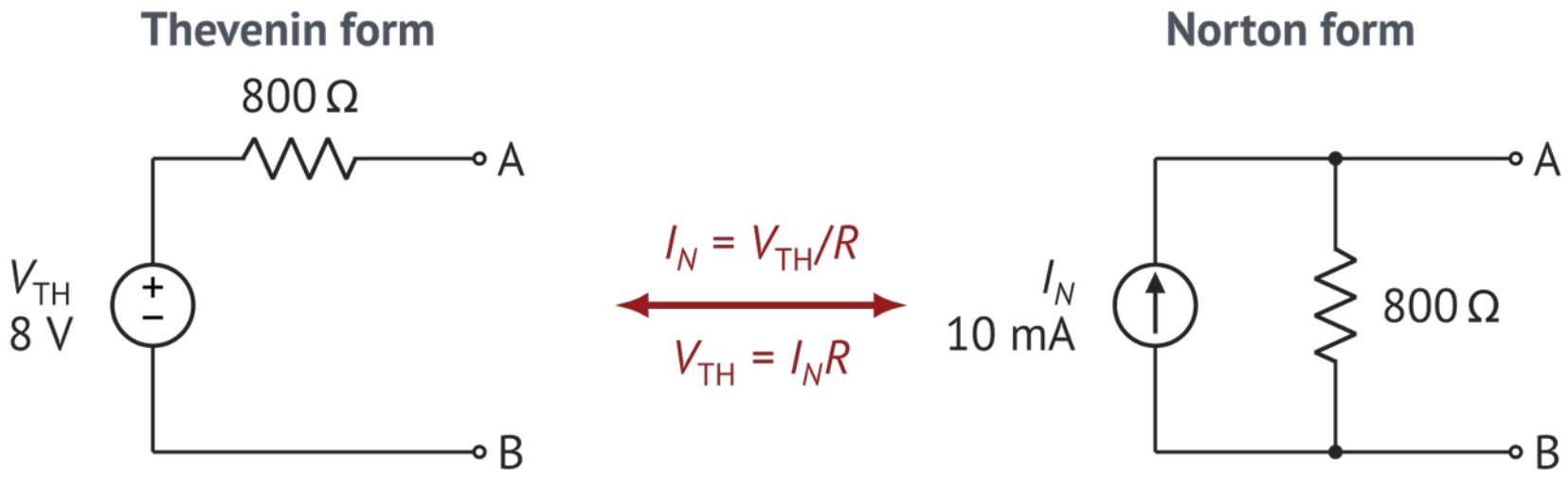
1. Replace the load with a wire from A to B
2. The wire is in parallel with R_2: 0 V across R_2, so no current in R_2
3. KVL around the outer loop: all 10 V is across R_1
4. I_N = 10 V / 1 kΩ = **10 mA** (0.01 A)
5. $R_N = R_{TH}$ = **800 Ω** (step 2 of the Thevenin example)

Check with the Thevenin values

$$I_N = \frac{V_{TH}}{R_{TH}} = \frac{8\,\text{V}}{800\,\Omega} = 10\,\text{mA}$$

Norton example: source transformation

A voltage source with a series resistor converts to a current source with the same resistor in parallel.



EXAMPLE

Same load, same answer. Attach R_3 = 2 kΩ to the Norton form; it is a current divider:

1. I_L = 10 mA × 0.8/(0.8 + 2) = **2.86 mA**
2. V_L = 2.86 mA × 2 kΩ = **5.71 V**, as in the Thevenin example

Source transformation

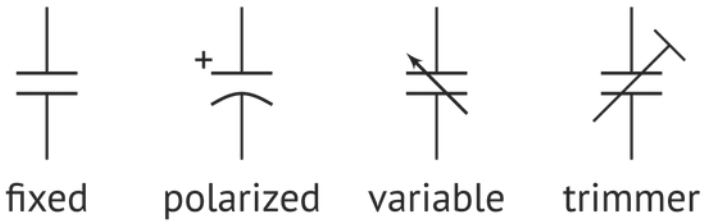
$$I_N = \frac{V_{TH}}{R_{TH}}, \qquad R_N = R_{TH}$$

8 V and 800 Ω $\iff$ 10 mA and 800 Ω

Capacitors

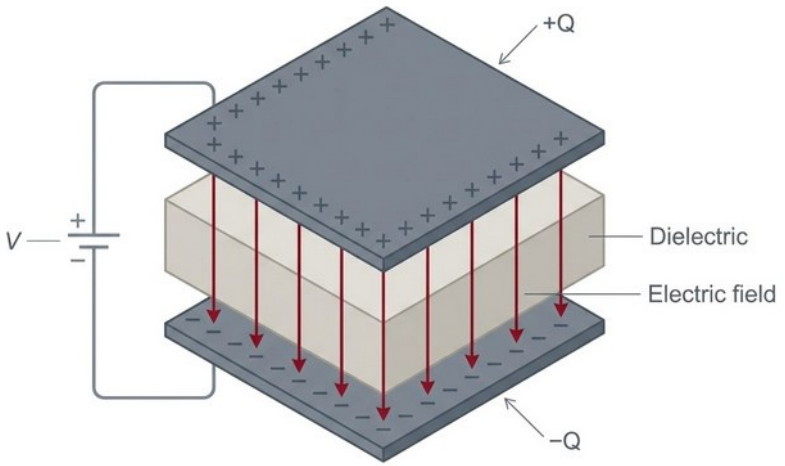
A capacitor stores charge on two conductors separated by an insulator, in proportion to the voltage.

Capacitor symbols



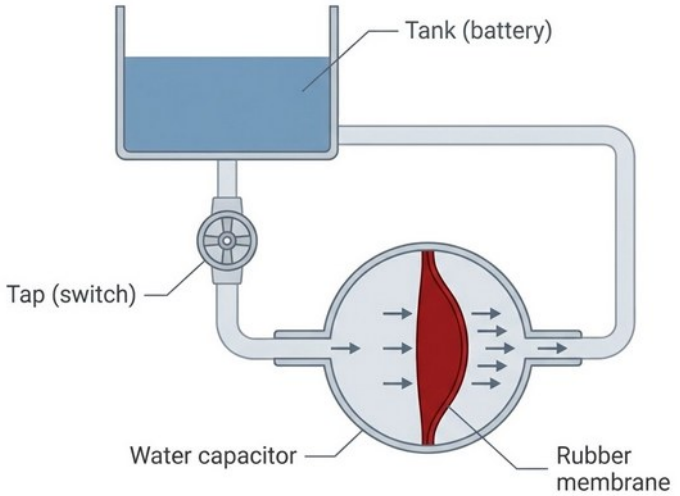
- Unit: farad (F); practical values pF to mF
- Polarized (electrolytic) parts have a + lead

Parallel-plate capacitor



+Q and −Q on the plates; the electric field in the dielectric stores the energy.

Water analogy



A rubber membrane in the pipe stretches until it pushes back as hard as the pump.

Capacitor relations

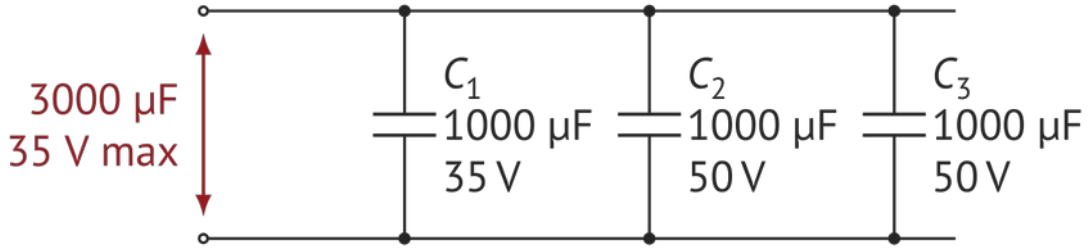
$$Q = CV \qquad C = \frac{\varepsilon A}{d} \qquad i = C\frac{dv}{dt} \qquad E = \tfrac{1}{2}CV^2$$

- Constant v: i = 0, an **open circuit** at DC
- The voltage cannot jump instantly

Capacitors in parallel and in series

Parallel capacitances add; in series their reciprocals add. These are the resistor rules swapped.

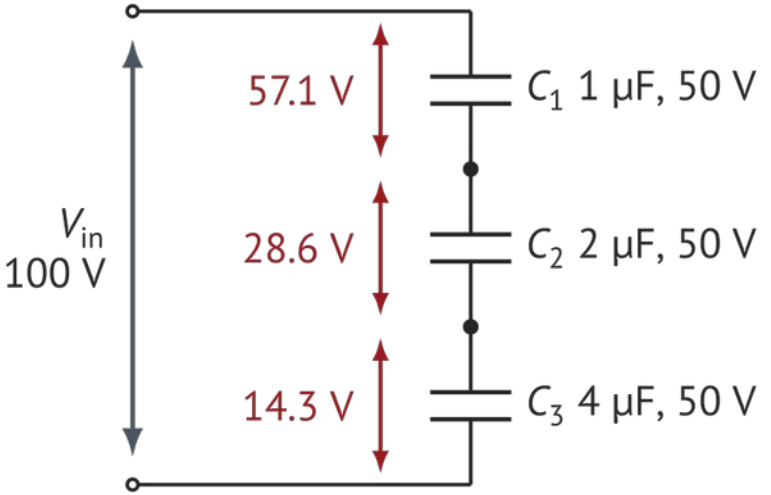
In parallel: more capacitance



$$C_{par} = C_1 + C_2 + C_3 = 3000\ \mu F$$

- Same voltage on each; the lowest rating (35 V) limits the bank

In series: higher voltage, less capacitance



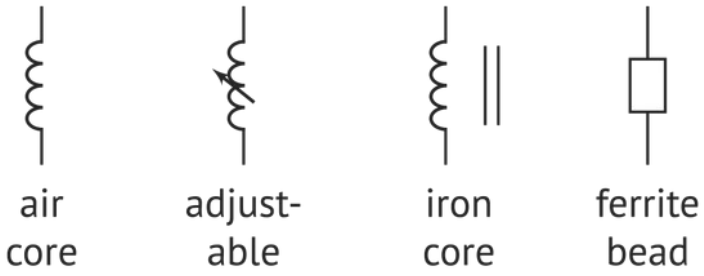
$$\frac{1}{C_{ser}} = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} \Rightarrow C_{ser} = 0.571\ \mu F$$

- Same charge (57.1 µC) on each, $V_k = Q/C_k$: the smallest C takes the most voltage
- C_1 is over its 50 V rating: the stack is safe only to 87.5 V, not 150 V

Inductors

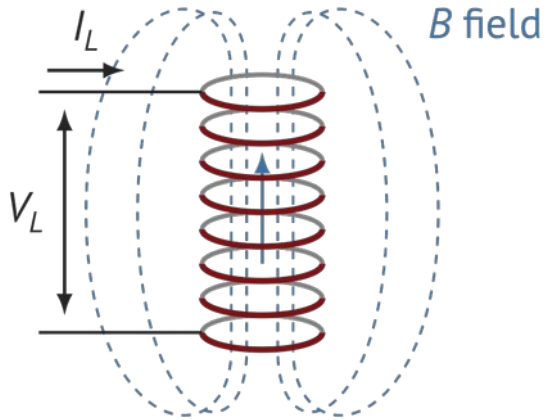
An inductor is a coil of wire; it stores energy in its magnetic field and opposes changes in current.

Inductor symbols



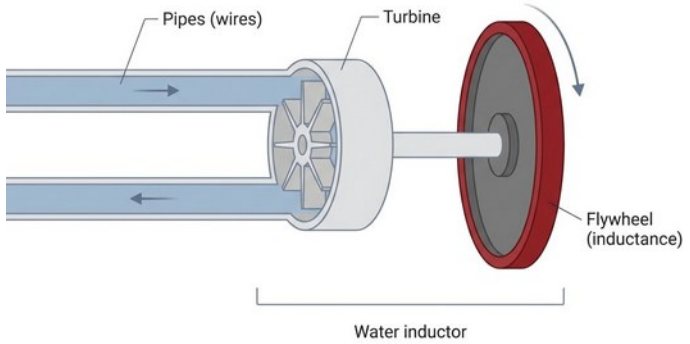
- Unit: henry (H); practical values nH to H
- An iron core raises L for the same coil

Air-core inductor



Current in the coil makes a magnetic field B; changing the current changes B, which induces V_L.

Water analogy



A turbine on a heavy flywheel resists spin-up, then keeps the water moving.

Inductor relations

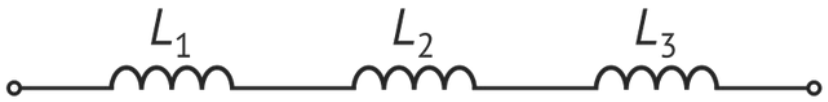
$$V_L = L\frac{dI_L}{dt} \qquad I_L = \frac{1}{L}\int V_L\,dt \qquad E = \tfrac{1}{2}LI_L^2$$

- Constant I: v = 0, a **short circuit** at DC
- The current cannot jump instantly

Inductors in series and in parallel

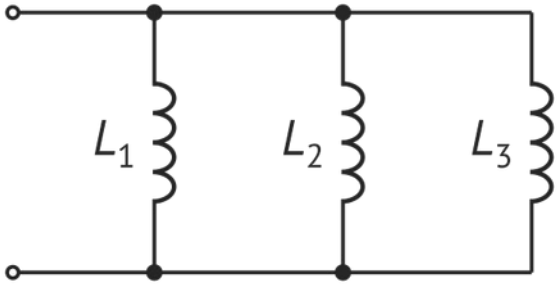
Inductors combine like resistors: series inductances add, and in parallel the reciprocals add.

In series



$$L_{ser} = L_1 + L_2 + L_3$$

In parallel



$$\frac{1}{L_{par}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

Series and parallel rules side by side

Element	Series	Parallel
Resistor R	add	reciprocals add
Inductor L	add	reciprocals add
Capacitor C	reciprocals add	add

EXAMPLE

L_1 = 1 mH, L_2 = 2 mH, L_3 = 4 mH

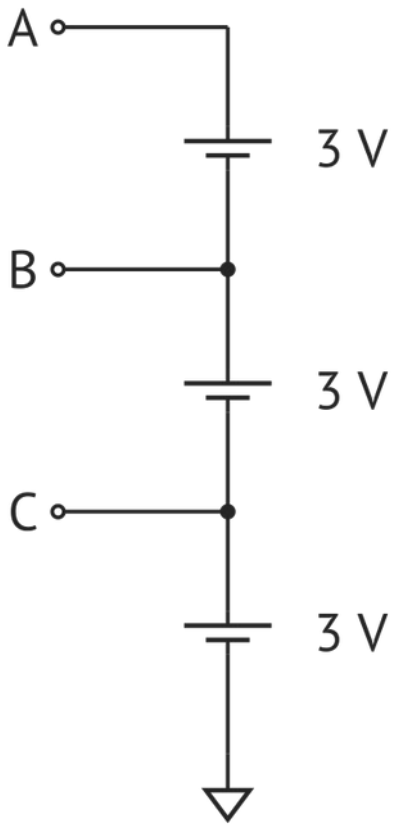
- Series: 1 + 2 + 4 = **7 mH**
- Parallel: 1/(1 + 1/2 + 1/4) = **0.571 mH**, below the smallest

Problem 1

Three 3 V cells are stacked on top of ground. What voltage does each node sit at?

PRACTICE

Given the circuit with nodes **A**, **B**, and **C**, each connected to **3 V** sources as shown, determine the voltages V_A, V_B, and V_C at the nodes with respect to ground.

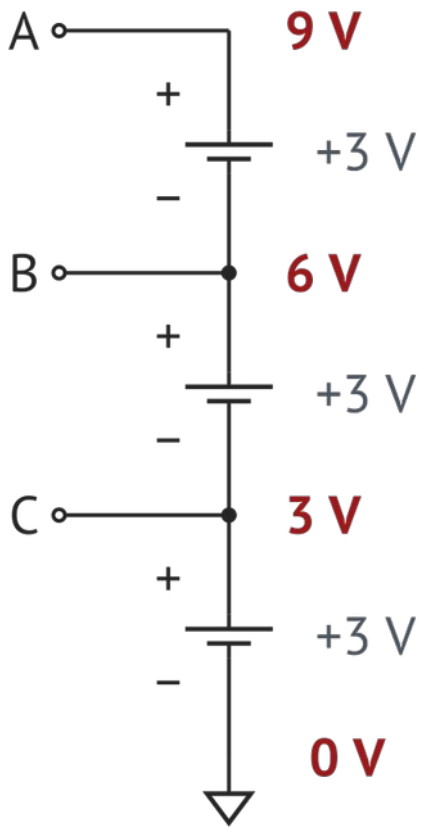


Questions to ask first

- Is there a closed path for current?
- Which plate of each cell is the + terminal?
- Where is 0 V?

Problem 1: solution

No current flows, so each cell simply adds 3 V to the node below it: 3 V, 6 V, 9 V.



SOLUTION

1. No closed loop, so no current flows; each cell only shifts the voltage
2. Start at ground (0 V) and walk up. The long plate is +, so each cell adds 3 V
3. $V_C = 0 + 3 = \mathbf{3\ V}$, $V_B = 3 + 3 = \mathbf{6\ V}$, $V_A = 6 + 3 = \mathbf{9\ V}$
4. A 9 V battery is built the same way: six 1.5 V cells in series

Answer

$$V_A = 9\ \text{V}$$
$$V_B = 6\ \text{V}$$
$$V_C = 3\ \text{V}$$

TRY IT

Flip the middle cell. What are V_A, V_B, and V_C now?

Recap: KCL

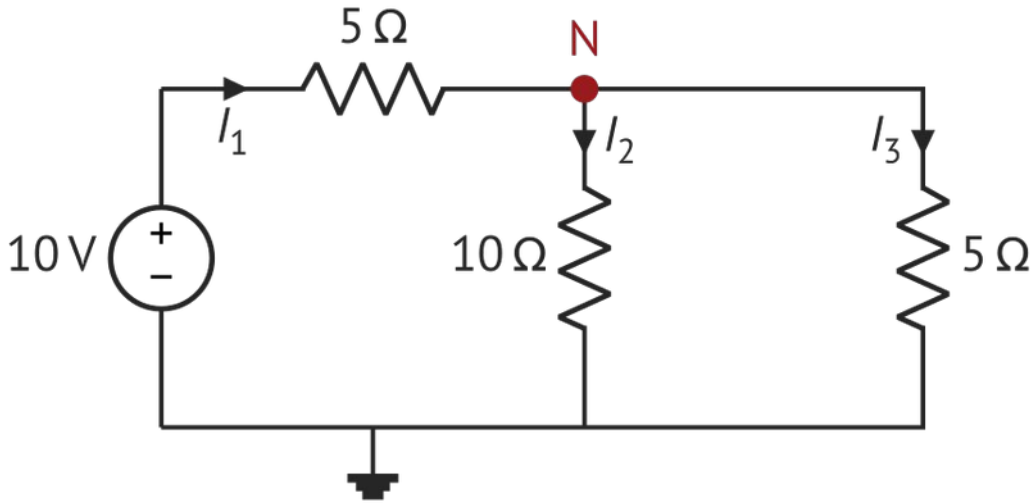
The current into a node equals the current out of it.

Kirchhoff's current law

$$\sum_{\text{in}} I = \sum_{\text{out}} I$$

- Charge cannot pile up or vanish at a node
- Pick a sign: leaving +, entering −; be consistent
- Current in a resistor = $(V_{start} - V_{end}) / R$
- One KCL equation per unknown node voltage

Watch out: a negative answer is fine; the current just flows against your arrow.



EXAMPLE

1. KCL at N: $(10 - V_N)/5 = V_N/10 + V_N/5$
2. $20 = 5\ V_N$, so $V_N =$ **4 V**
3. $I_1 =$ **1.2 A** $= I_2 + I_3 = 0.4 + 0.8$ ✓

Recap: KVL

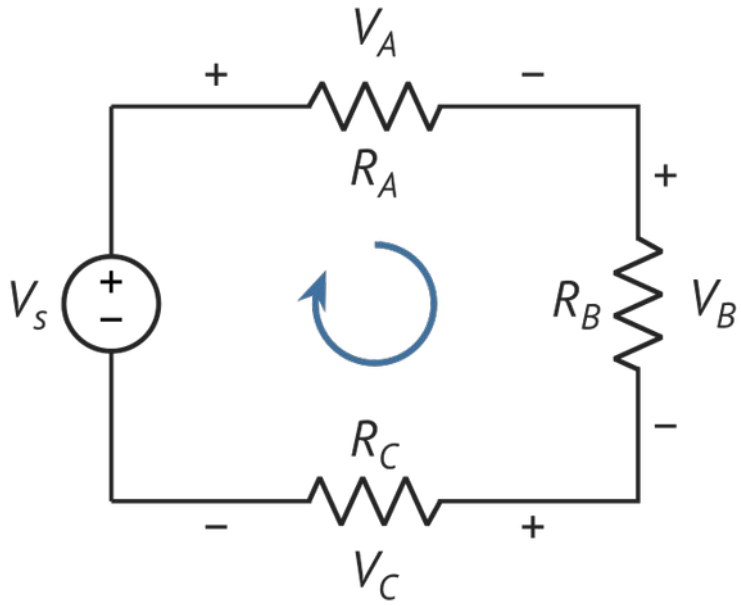
The voltages around any closed loop add to zero.

Kirchhoff's voltage law

$$\sum_{\text{loop}} V_k = 0$$

- Energy is conserved: every rise is paid back by drops
- Enter at +: subtract (downhill); enter at −: add (uphill)
- Clockwise here: $V_s - V_A - V_B - V_C = 0$
- So $V_s = V_A + V_B + V_C$

Watch out: trace each polarity; a flipped source subtracts instead of adding.



EXAMPLE

$V_s = 12$ V; $R_A, R_B, R_C = 5, 4, 3$ kΩ

1. I = 12 V / 12 kΩ = 1 mA
2. V_A, V_B, V_C = **5, 4, 3 V**; 5 + 4 + 3 = 12 ✓

Recap: voltage divider

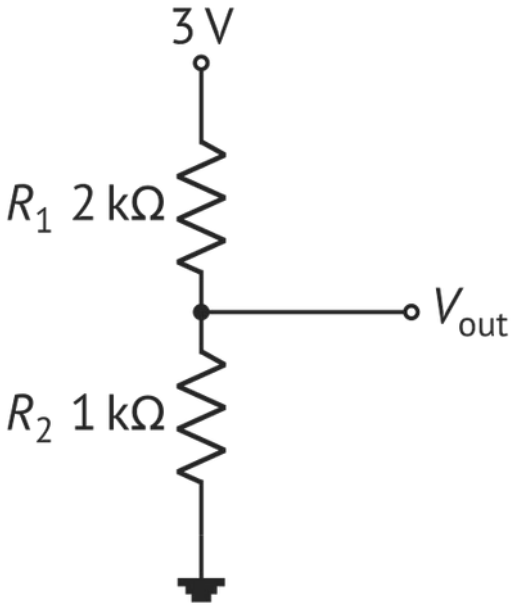
Two series resistors split a voltage in proportion to their resistances.

Divider rule

$$V_2 = V_s \frac{R_2}{R_1 + R_2}$$

- Same current through both resistors
- Larger resistor, larger share of the voltage
- Only the ratio $R_1 : R_2$ matters
- Used for ADC scaling, sensors and the touch panel

Watch out: the rule assumes nothing draws current from V_{out}; a load changes it.



EXAMPLE

1. I = 3 V / (2 kΩ + 1 kΩ) = **1 mA**
2. V_{R1} = 3 × 2/3 = **2 V**, V_{R2} = V_{out} = **1 V**
3. Check: 2 V + 1 V = 3 V ✓

Recap: current divider

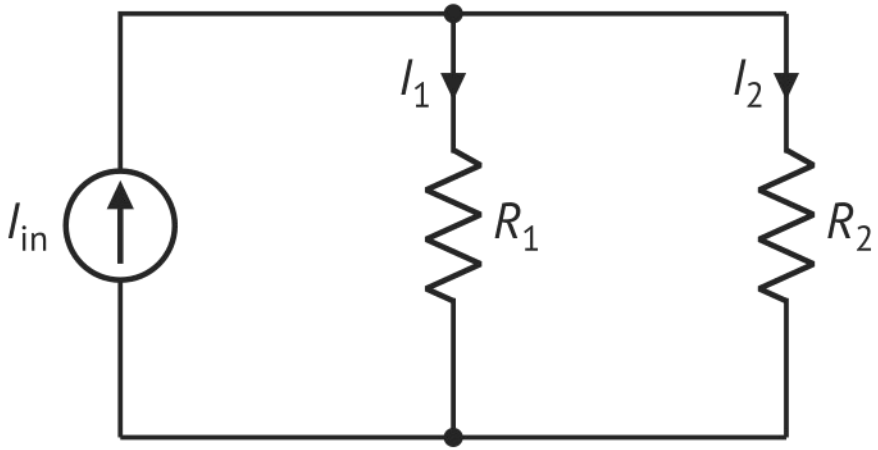
Parallel resistors split a current; the smaller resistor takes more.

Current divider rule

$$I_1 = I_{in} \frac{R_2}{R_1 + R_2}$$

- Same voltage across both resistors
- Each current has the **other** resistor on top
- Smaller resistor, larger share of the current
- Many branches: $I_k = I_{in}\, R_{par} / R_k$

Watch out: check that the smaller resistor got the larger current.



EXAMPLE

I_{in} = 10 mA into R_1 = 2 kΩ ‖ R_2 = 3 kΩ

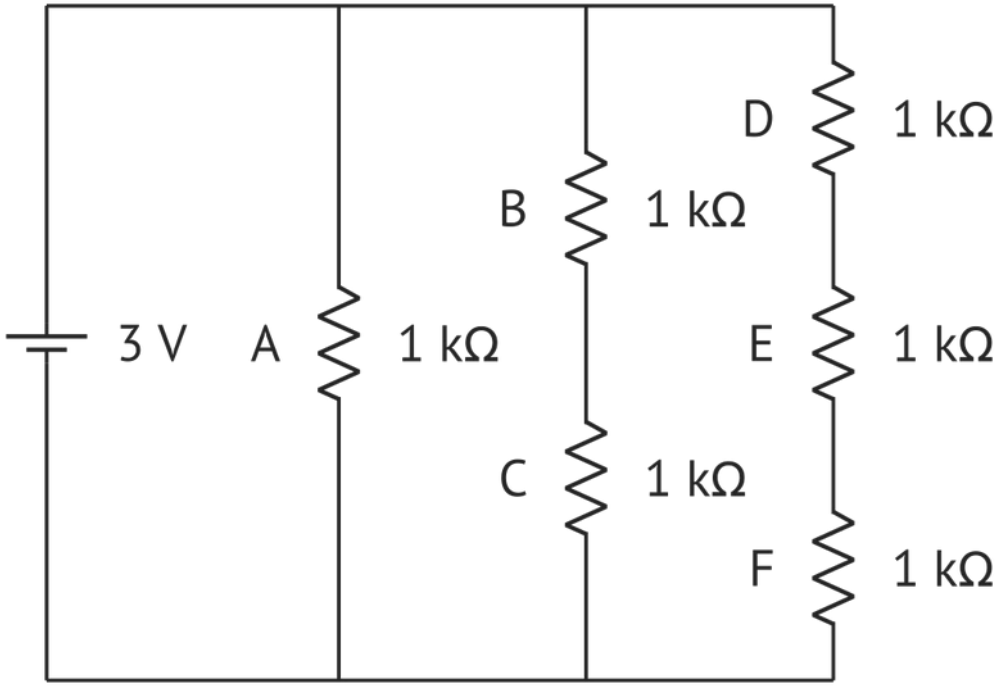
1. $I_1 = 10 \times 3/5 =$ **6 mA**, $I_2 = 10 \times 2/5 =$ **4 mA**
2. Check: 6 mA × 2 kΩ = 4 mA × 3 kΩ = 12 V ✓

Problem 2

Six 1 kΩ resistors in three branches across one 3 V battery. Find every voltage and current.

PRACTICE

Given the circuit with six resistors connected to a **3 V** voltage source as shown, find the voltages across the resistors (V_A, V_B, V_C, V_D, V_E, and V_F) and the currents through the resistors (I_A, I_B, I_C, I_D, I_E, and I_F).

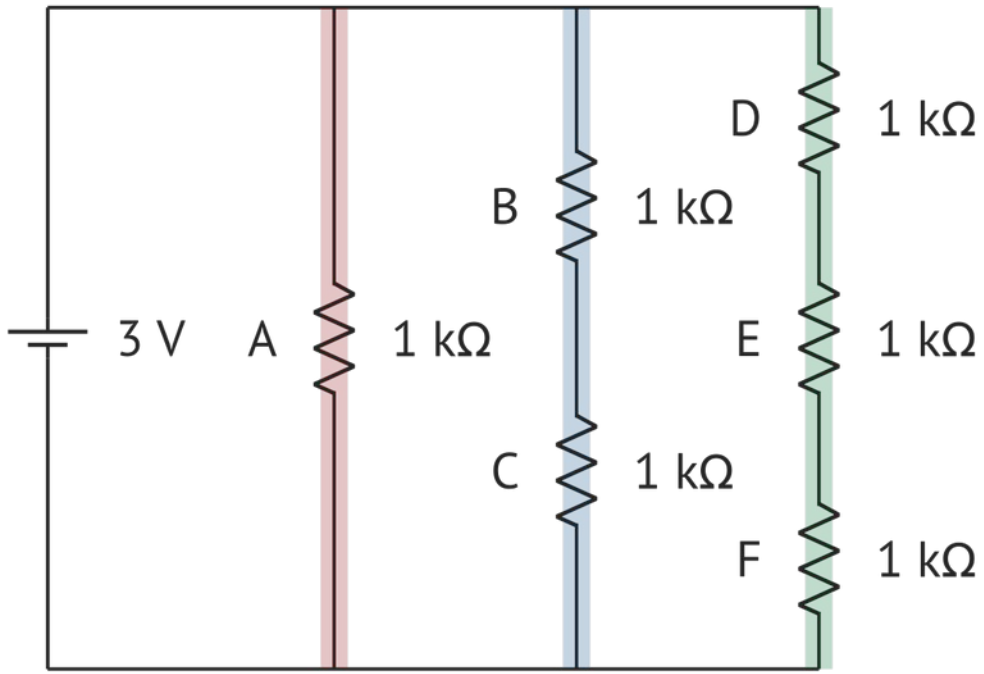


Questions to ask first

- How many nodes are there?
- Which resistors carry the same current?
- What voltage is across each branch?

Problem 2: solution

Each branch connects straight across the battery, so every branch has the full 3 V across it.



SOLUTION

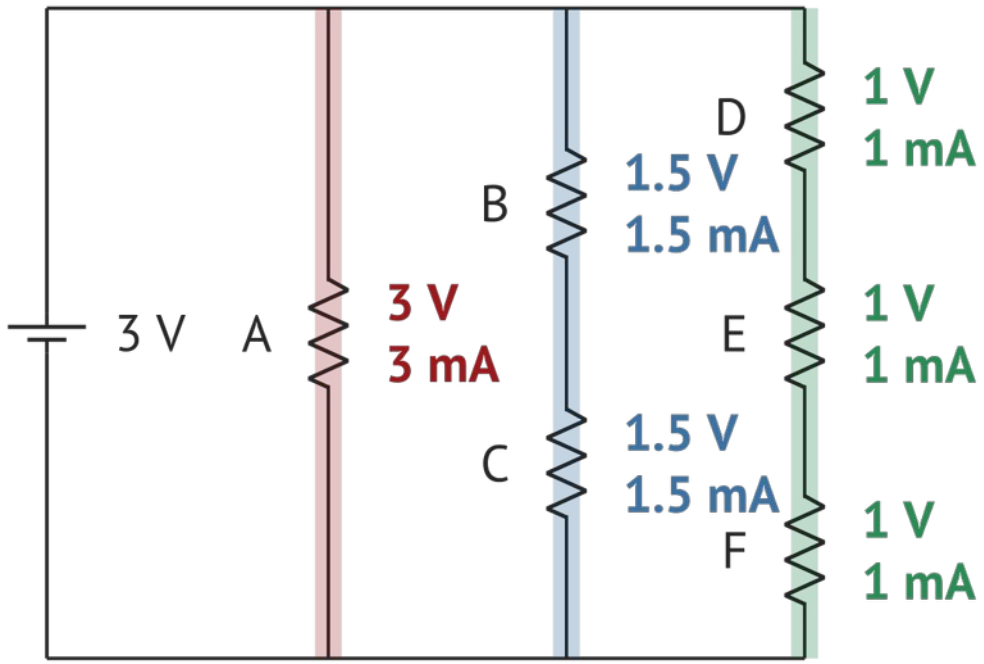
1. The top wire is one node (+3 V); the bottom wire is another (0 V)
2. Each branch spans those two nodes: 3 V across every branch
3. **Branch A**: V_A = **3 V**, I_A = 3 V / 1 kΩ = **3 mA**
4. **Branch B-C** (2 kΩ in series): $I_B = I_C$ = 3 V / 2 kΩ = **1.5 mA**, so $V_B = V_C$ = **1.5 V**

One branch at a time

$$I_{branch} = \frac{3\ V}{R_{branch}} \qquad V_k = I_{branch} \times 1\ k\Omega$$

Problem 2: solution (continued)

Equal resistors in series share the branch voltage evenly; the battery supplies all three branch currents.



SOLUTION

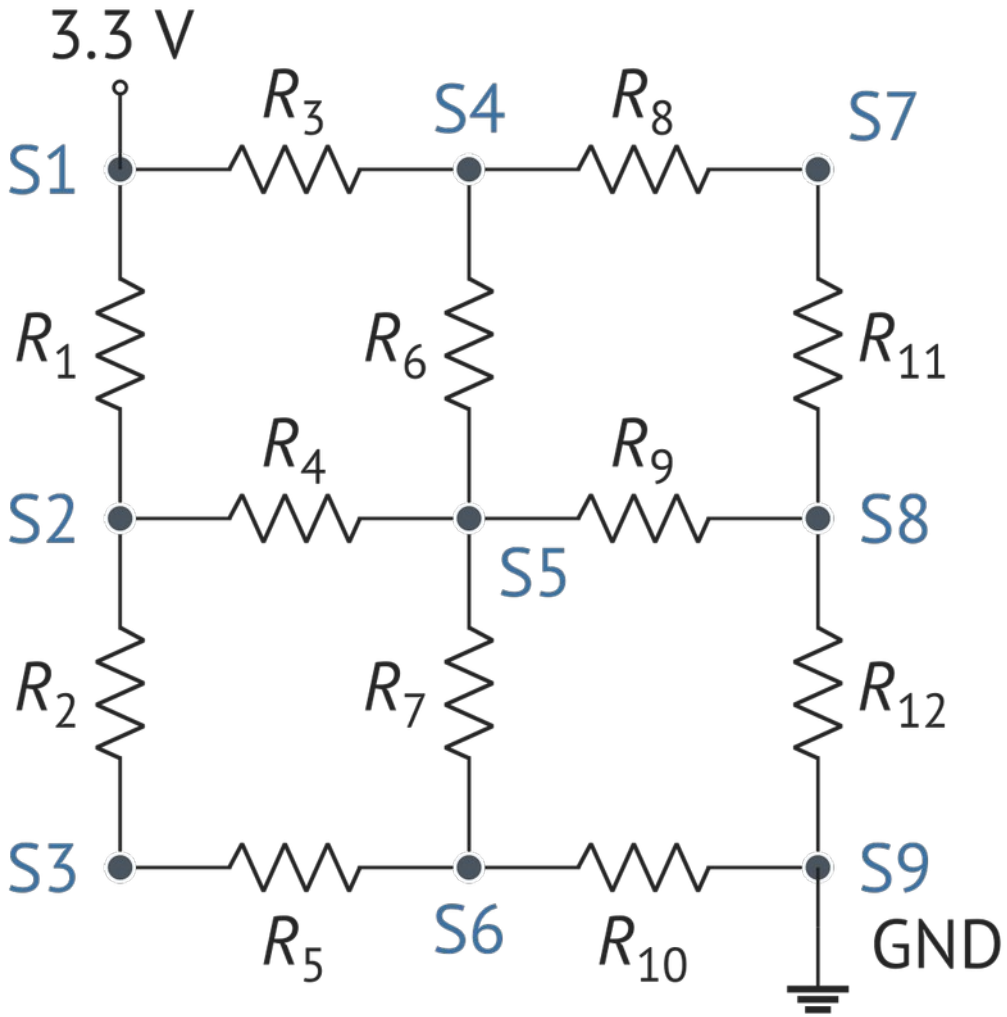
1. **Branch D-E-F** (3 kΩ): $I_D = I_E = I_F = 3\ V / 3\ k\Omega = \mathbf{1\ mA}$
2. Each drops 1 mA × 1 kΩ: $V_D = V_E = V_F = \mathbf{1\ V}$
3. KVL check: 1.5 + 1.5 = 1 + 1 + 1 = 3 V in every branch
4. KCL at the top node: $I_{battery} = 3 + 1.5 + 1 = \mathbf{5.5\ mA}$
5. The network looks like 3 V / 5.5 mA = **545 Ω**, less than the smallest branch

Equivalent resistance

$$R_{eq} = \frac{1\ k\Omega}{1 + \frac{1}{2} + \frac{1}{3}} = 0.545\ k\Omega = 545\ \Omega$$

Problem 3

The touchpad grid again, with every resistor equal to R: find all nine node voltages.



PRACTICE

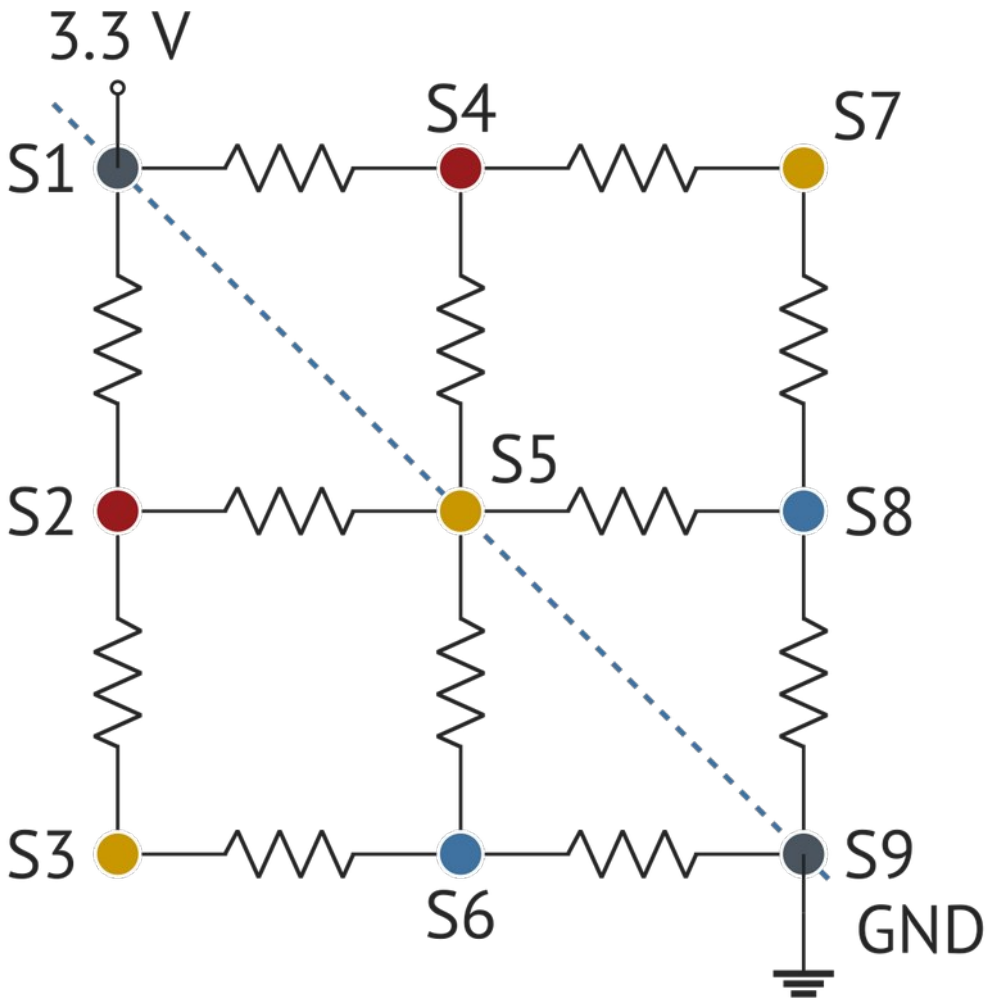
Given the circuit with nodes S1 to S9, S1 is connected to **3.3 V** and S9 to GND (0 V). Determine the voltages S1 to S9 with respect to ground if R_1 to R_{12} = R.

Hint

Before writing 21 equations, look at the symmetry of the grid. Which nodes must have equal voltages? What changes if you turn the grid upside down and swap 3.3 V with ground?

Problem 3: solution

Two symmetries of the grid pin three node voltages and leave a single unknown, S2.



SOLUTION

1. **Mirror** across the S1-S5-S9 diagonal: S2 = S4, S3 = S7, S6 = S8
2. **Rotate 180° and swap 3.3 V with ground**: $S_k + S_{10-k}$ = 3.3 V, so S5 = 1.65 V and S8 = 3.3 − S2
3. Combine: S3 = S7 = 3.3 − S3, so S3 = S7 = 1.65 V

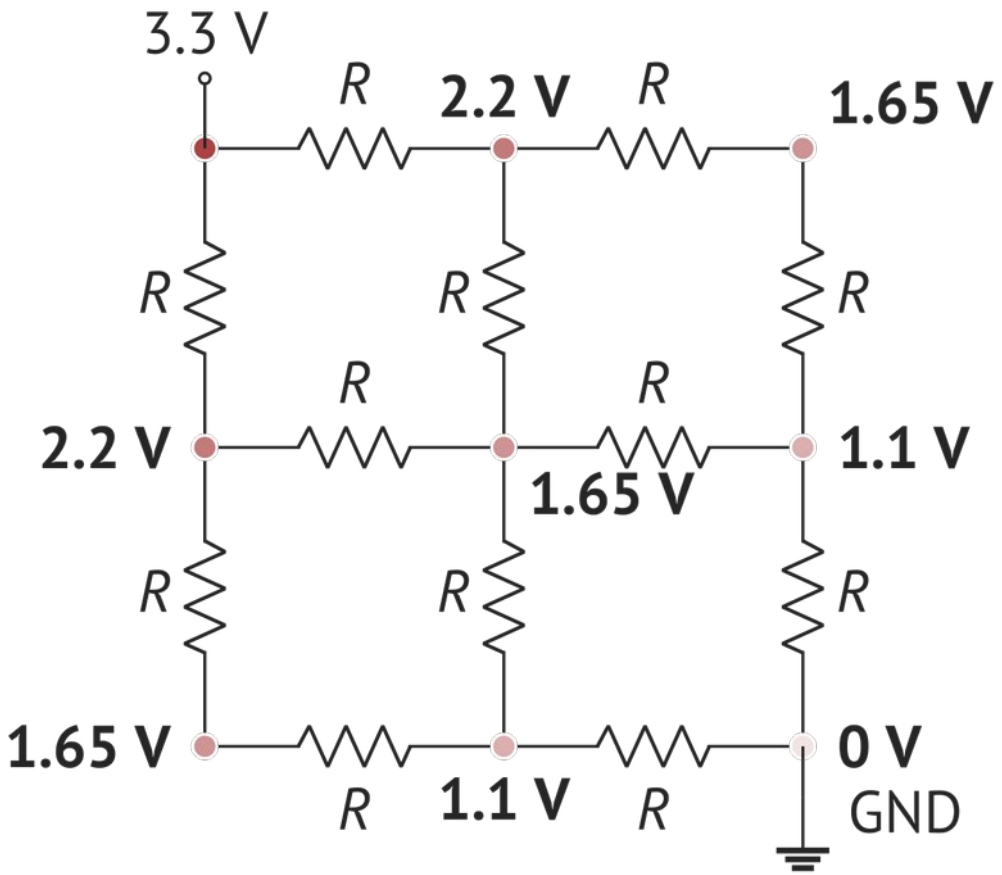
Known so far (volts)

$$S_1 = 3.3, \quad S_3 = S_5 = S_7 = 1.65, \quad S_9 = 0$$
$$S_4 = S_2, \quad S_6 = S_8 = 3.3 - S_2$$

Only S2 is still unknown: one KCL equation finds it.

Problem 3: solution (continued)

One KCL equation at S2 finishes the job, and R cancels: the voltages do not depend on R.



4. KCL at S2 (neighbors S1, S3, S5; currents leaving):

$$\frac{S_2 - 3.3}{R} + \frac{S_2 - 1.65}{R} + \frac{S_2 - 1.65}{R} = 0 \quad \Rightarrow \quad S_2 = 2.2\ \text{V}$$

Answer (volts, for any R)

S1	S2	S3	S4	S5	S6	S7	S8	S9
3.3	2.2	1.65	2.2	1.65	1.1	1.65	1.1	0

- Currents: 1.1 V/R in R_1, R_3, R_{10}, R_{12}; 0.55 V/R in the rest
- Rail current 2.2 V/R: the grid is **1.5 R** from S1 to S9
- R = 100 Ω gives Network 2's 11, 5.5 and 22 mA